Hamiltonian Methods for the Study of Polarized Proton Beam Dynamics in Accelerators and Storage Rings

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Abstract

The equations of classical spin-orbit motion can be extended to a **Hamiltonian system** in 9-dimensional phase space by introducing a **coupled spin-orbit Poisson bracket** (18) and a **Hamiltonian function** (20). After this extension and by establishing connections between initial and extended systems it becomes possible to apply the **methods of the theory of Hamiltonian systems** to the study of polarized particle beam dynamics in circular accelerators and storage rings. Some of those methods have been implemented in the **computer code FORGET-ME-NOT** [1, 2].

Contents

1 Introduction

We began writing this paper in the summer of 1994 year following the suggestion of D.Barber to summarize our results of the investigation of the dynamics of unpolarized and polarized proton beams, which we obtained during 1991-1993 and which were only partly available and only in the very compressed form of conference proceedings.

After starting this work it became clear that it would be better to separate the available material into several papers, and for the first one we chose those results which can be incorporated together with help of the research technique which we call **Hamiltonian extension of the equations of classical spin-orbit motion**. Such important topics as the computation and analysis of the one-turn Taylor maps for spin and orbit motion, spin dependent invariant functions and their connection with the Derbenev-Kondratenko vector, the normal form algorithm for equations of spin motion in the SU(2) representation and, of course, many more practical subjects related to the question how to preserve the polarization of a particle beam during acceleration will not appear in the present paper and will be published later.

For various reasons the publication has been prolonged for almost three years, and now we are glad to be free from this, but, at the same time, we are not too enthusiastic about the necessity to write up the remaining unpublished results.

1.1 The Classical Equations of Spin-Orbit Motion

The quasi-classical description of the motion of a relativistic nonradiating point particle with spin in accelerators and storage rings includes the equations of orbit motion which we write in the Hamiltonian form

$$\frac{d\vec{q}}{dt} \;=\; \frac{\partial H_{orbt}}{\partial \vec{p}}, \qquad\qquad \frac{d\vec{p}}{dt} \;=\; -\,\frac{\partial H_{orbt}}{\partial \vec{q}} \tag{1}$$

and the Thomas-BMT equation [3, 4] for the classical spin vector $\vec{s}$

$$\frac{d\vec{s}}{dt} \;=\; \left[\, \vec{W} \times \vec{s}\, \right] \tag{2}$$

Here

$$H_{orbt} \;=\; c\,\sqrt{\vec{\pi}^{\,2} \;+\; m_0^2\, c^2} \;+\; e\,\Phi$$

$$\vec{W} \;=\; -\frac{e}{m_0\gamma c}\left((1+\gamma G)\,\vec{\mathcal{B}} \;-\; \frac{G\left(\vec{\pi}\cdot\vec{\mathcal{B}}\right)\vec{\pi}}{m_0^2c^2(1+\gamma)} \;-\; \frac{1}{m_0c}\left(G+\frac{1}{1+\gamma}\right)\left[\vec{\pi}\times\vec{\mathcal{E}}\right]\right)$$

and t is the time. The vectors $\vec{q} = (q_1, q_2, q_3)$ and $\vec{p} = (p_1, p_2, p_3)$ are canonical position and momentum variables, and $\vec{s} = (s_1, s_2, s_3)$ is the classical spin vector of length $\hbar/2$. The parameters e and m_0 are the charge and the rest mass of the particle, c is the velocity of light, $G = (g-2)/2$ which quantifies the anomalous spin g factor, γ is the Lorentz factor, $\vec{\pi}$ is kinetic momentum vector, $\vec{\mathcal{E}}$ and $\vec{\mathcal{B}}$ are the electric and magnetic fields, and $\vec{A}$ and Φ are the vector and scalar potentials.

$$\vec{\mathcal{B}} \;=\; \mathrm{curl}_{\,\vec{q}}\,\vec{A} \tag{3}$$

$$\vec{\mathcal{E}} \;=\; -\,\mathrm{grad}_{\,\vec{q}}\,\Phi \;-\; \frac{1}{c}\,\frac{\partial\vec{A}}{\partial t} \tag{4}$$

$$\vec{\pi} \;=\; \vec{p} \;-\; \frac{e}{c}\,\vec{A}$$

$$\gamma \;=\; \frac{H_{orbt}-e\Phi}{m_0c^2} \;=\; \sqrt{1 \;+\; \left(\frac{\vec{\pi}}{m_0c}\right)^2} \tag{5}$$

Later on we will refer to the system (1)-(2) as the **triangular system** (the equations of spin motion contain the orbital variables but the evolution of orbital variables does not depend on the spin degree of freedom).

Remark: In the Hamiltonian picture of orbital motion we cannot define the Lorentz factor (5), in the usual way, in terms of the absolute value v of the particle velocity $\vec{v}$

$$\gamma \;=\; \frac{1}{\sqrt{1 \;-\; (v/c)^2}}, \qquad v^2 \;=\; \vec{v}\cdot\vec{v}$$

but instead need to express it through canonical variables.

2 The Hamiltonian Extension of the Equations of Classical Spin-Orbit Motion

The Poisson bracket lies at the basis of the modern point of view of the Hamiltonian formalism. The idea of axiomatic introduction of the bracket very likely belongs to Dirac. In this section we recall the necessary definitions from the theory of Hamiltonian systems and introduce coupled spin-orbit Poisson brackets which give us the canonical extension of the equations of classical spin-orbit motion.

2.1 Poisson Bracket

Let M be a finite-dimensional manifold (phase space) and $C^\infty(M)$ be the linear space of smooth functions $f : M \to R$. The binary operation

$$\{*,\, *\} \,:\, C^\infty(M) \times C^\infty(M) \;\to\; C^\infty(M)$$

called the **Poisson bracket** satisfies the relations

bilinearity : $\{\lambda f \,+\, \mu g,\, h\} \;=\; \lambda\,\{f,\, h\} \,+\, \mu\,\{g,\, h\}$

antisymmetry : $\{f,\, g\} \;=\; -\{g,\, f\}$

Leibnitz rule : $\{f,\, g\cdot h\} \;=\; \{f,\, g\}\cdot h \,+\, g\cdot\{f,\, h\}$

Jacobi identity : $\{\{f,\, g\},\, h\} \,+\, \{\{h,\, f\},\, g\} \,+\, \{\{g,\, h\},\, f\} \;=\; 0$

where $\lambda,\ \mu$ are arbitrary constants.

Let $\vec{z}$ be local coordinates on a manifold M. From the bilinearity and Leibnitz rule it follows that for any fixed function h, the linear operator

$$\{h,\, *\} \,:\, C^\infty(M) \;\to\; C^\infty(M)$$

is a differentiation and hence may be represented in the form (see, for example [14, 15])

$$\{h,\, *\} \;=\; A_i^h(\vec{z})\,\frac{\partial *}{\partial z_i} \tag{6}$$

(here and further on the summation over repeated indices is implied). From (6) one finds by direct substitution of functions z_i that

$$A_i^h(\vec{z}) \;=\; \{h,\; z_i\} \tag{7}$$

Comparing (6) and (7) we obtain

$$\{h,\; *\} \;=\; \{h,\; z_i\}\, \frac{\partial *}{\partial z_i} \tag{8}$$

Consider the Poisson bracket $\{f,\; g\}$. By successively applying (8) to the expressions $\{f,\, *\}$ and $\{*,\, z_j\}$ we obtain

$$\{f,\; g\} \;=\; \{f,\; z_j\}\, \frac{\partial g}{\partial z_j} \;=\; \{z_i,\; z_j\}\, \frac{\partial f}{\partial z_i}\, \frac{\partial g}{\partial z_j} \tag{9}$$

Now we introduce the skew-symmetric matrix $\hat{J} = (\{z_i,\; z_j\})$ and represent the Poisson bracket (9) in the form

$$\{f,\; g\} \;=\; \mathrm{grad}_{\,\vec{z}}\, f \cdot \hat{J}\; \mathrm{grad}_{\,\vec{z}}\, g \tag{10}$$

Thus in a fixed local coordinate system $\vec{z}$, the Poisson bracket is completely defined if we know the values $\hat{J}_{ij} \;=\; \{z_i,\; z_j\}$ as functions of $\vec{z}$.

The matrix $\hat{J}$ is called **the structural matrix** of a Poisson bracket.

2.2 Hamiltonian Dynamical Systems

Again let $\vec{z}$ be local coordinates on a manifold M . **Hamiltonian systems** by definition have the form:

$$\frac{d\vec{z}}{d\tau} \;=\; \{\vec{z},\; H\} \tag{11}$$

where $H \;=\; H(\tau,\, \vec{z})$ is an arbitrary function (possibly depending explicitly on τ[1]), called the **Hamiltonian**. Using the representation of Poisson bracket (10) we can rewrite equations (11) as

$$\frac{d\vec{z}}{d\tau} \;=\; \hat{J}(\vec{z})\; \mathrm{grad}_{\,\vec{z}}\, H \tag{12}$$

[1]Until now the equations of classical spin-orbit motion (1)-(2) are written using the time t as independent variable, but later on we will change this independent variable to be a path length along the design orbit z. This is the reason for us, here and further on, to formulate general results, which are not sensitive to the specific form of the variables and Hamiltonians used, we denote the independent variable as τ.

Let $h = h(\vec{z})$ be an arbitrary function and $\vec{z}(\tau)$ be the solution of the system (11) with initial condition $\vec{z}(\tau_0) = \vec{z}_0$. It is clear that the derivative of the function $h(\tau) = h(\vec{z}(\tau))$ for the Hamiltonian system (11) has the form

$$\frac{dh}{d\tau} = \{h, H\} \tag{13}$$

From (13) we can obtain the expansion of $h(\tau)$ with respect to a small $\triangle\tau = \tau - \tau_0$

$$h(\tau) = h(\tau_0) + \triangle\tau \{h, H\}(\tau_0) + O(\triangle\tau^2) \tag{14}$$

Writing $f(\tau)$ and $g(\tau)$ instead $h(\tau)$ in (14) and calculating their Poisson brackets with respect to the variables $\vec{z}_0$, we see that

$$\{f, g\}(\tau) = \{f, g\}(\tau_0) +$$

$$+ \triangle\tau (\{f, \{g, H\}\}(\tau_0) + \{\{f, H\}, g\}(\tau_0)) + O(\triangle\tau^2) \tag{15}$$

Applying Jacobi identity to the terms of order $\triangle\tau$ in (15) one finds

$$\{f, g\}(\tau) = \{f, g\}(\tau_0) + \triangle\tau \{\{f, g\}, H\}(\tau_0) + O(\triangle\tau^2)$$

or, equivalently

$$\frac{\{f, g\}(\tau) - \{f, g\}(\tau_0)}{\triangle\tau} = \{\{f, g\}, H\}(\tau_0) + O(\triangle\tau) \tag{16}$$

The formula (16) in the limit when $\triangle\tau \to 0$ gives us the fundamental property of the solution of the Hamiltonian system:

The flow of a Hamiltonian system preserves the Poisson bracket:[2]

$$\frac{d}{d\tau}\{f, g\} = \{\{f, g\}, H\} \tag{17}$$

[2]We use the term ″preserve″ to mean that the algebraic form is not changed.

2.3 Hamiltonian Extension of the Equations of Classical Spin-Orbit Motion

We now introduce the Poisson bracket[3]

$$\{f(\vec{z}),\ g(\vec{z})\} \ = \ f_{\vec{q}} \cdot g_{\vec{p}} \ - \ f_{\vec{p}} \cdot g_{\vec{q}} \ + \ [\, f_{\vec{s}} \times g_{\vec{s}}\,] \cdot \vec{s} \tag{18}$$

in the 9-dimensional phase space $\vec{z} = (\vec{x}, \vec{s})$ of 6 orbital variables $\vec{x} = (\vec{q}, \vec{p})$ and 3 spin variables $\vec{s}$ and consider a Hamiltonian system of ordinary differential equations

$$\frac{d\vec{z}}{dt} \ = \ \{\vec{z},\ H\} \tag{19}$$

with the Hamiltonian function

$$H \ = \ H_{orbt}(t, \vec{x}) \ + \ \vec{W}(t, \vec{x}) \cdot \vec{s} \tag{20}$$

In the variables $\vec{q}$, $\vec{p}$ and $\vec{s}$ the system (19) can be written as

$$\frac{d\vec{q}}{dt} \ = \ \frac{\partial H_{orbt}}{\partial \vec{p}} \ + \ \frac{\partial \left(\vec{W} \cdot \vec{s}\,\right)}{\partial \vec{p}} \tag{21}$$

$$\frac{d\vec{p}}{dt} \ = \ -\frac{\partial H_{orbt}}{\partial \vec{q}} \ - \ \frac{\partial \left(\vec{W} \cdot \vec{s}\,\right)}{\partial \vec{q}} \tag{22}$$

$$\frac{d\vec{s}}{dt} \ = \ \left[\vec{W} \times \vec{s}\,\right] \tag{23}$$

and we will understand the equations (21)-(23) as the **Hamiltonian extension of the equations of classical spin-orbit motion** (1)-(2).

Note that the matrix $\hat{J}(\vec{z})$ for the spin-orbit Poisson bracket (18) has

[3]Note that when applied to the spin variables only, the Poisson bracket (18) gives the usual result $\{\, s_i,\ s_j \,\} \ = \ \epsilon_{ijk}\, s_k$.

the form

$$\hat{J}(\vec{z}) \;=\; \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & s_3 & -s_2 \\ 0 & 0 & 0 & 0 & 0 & 0 & -s_3 & 0 & s_1 \\ 0 & 0 & 0 & 0 & 0 & 0 & s_2 & -s_1 & 0 \end{pmatrix} \tag{24}$$

and is not a constant matrix (in contrast to the case of classical Poisson brackets), but depends on spin variables. The structural matrix (24) can also be written in the more compact block diagonal form

$$\hat{J}(\vec{z}) \;=\; \mathrm{diag}\,(J,\; J_s(\vec{s}))$$

where a 6×6 constant matrix J is the symplectic unit [12] and

$$J_s(\vec{s}\,) \;=\; \begin{pmatrix} 0 & s_3 & -s_2 \\ -s_3 & 0 & s_1 \\ s_2 & -s_1 & 0 \end{pmatrix}$$

2.4 Connection between the Triangular System and its Hamiltonian Extension

We now wish to point out some ways for establishing the connections between properties and solutions of the system (21)-(23) and the initial triangular system (1)-(2) (**truncation procedures**). We will do this **without ascribing any physical sense to the spin dependent members in the right sides of equations (21), (22)** [4].

Because the statements, which will be listed below, are connected not with the specific form of the spin-orbit Hamiltonian, but only with properties of the spin-orbit Poisson bracket, let us consider an arbitrary smooth Hamiltonian function possibly depending nonlinearly on the variables $\vec{s}$

$$H \;=\; H(\tau,\, \vec{z}) \;=\; H(\tau,\, \vec{x},\, \vec{s}) \tag{25}$$

[4] Note that some authors (see, for example [33, 34]) ascribe the spin dependent members on the right sides of equations (21)-(22) to a quasi-classical effect of the spin on the orbit motion, so that in that case no truncation procedures are needed.

and introduce, in correspondence to the Hamiltonian system

$$\frac{d\vec{x}}{d\tau} \;=\; J\ \mathrm{grad}_{\vec{x}}\, H, \qquad \frac{d\vec{s}}{d\tau} \;=\; J_s(\vec{s})\ \mathrm{grad}_{\vec{s}}\, H \tag{26}$$

the triangular truncated system defined as

$$\frac{d\vec{x}}{d\tau} \;=\; J\cdot(\mathrm{grad}_{\vec{x}}\, H)|_{\vec{s}=\vec{0}}\,, \qquad \frac{d\vec{s}}{d\tau} \;=\; J_s(\vec{s})\cdot(\mathrm{grad}_{\vec{s}}\, H)|_{\vec{s}=\vec{0}} \tag{27}$$

a) If

$$\vec{z}(\tau,\,\tau_0,\,\vec{z}_0) \;=\; (\vec{x}(\tau,\,\tau_0,\,\vec{x}_0,\,\vec{s}_0),\ \vec{s}(\tau,\,\tau_0,\,\vec{x}_0,\,\vec{s}_0))$$

is the solution of (26) which passes through the point $\vec{z}_0 \;=\; (\vec{x}_0,\ \vec{s}_0)$ when $\tau \;=\; \tau_0$, then

$$\vec{z}_*(\tau,\,\tau_0,\,\vec{z}_0) \;=\; (\vec{x}_*(\tau,\,\tau_0,\,\vec{x}_0),\ \vec{s}_*(\tau,\,\tau_0,\,\vec{x}_0,\,\vec{s}_0))$$

where

$$\vec{x}_*(\tau,\,\tau_0,\,\vec{x}_0) \;=\; \vec{x}(\tau,\,\tau_0,\,\vec{x}_0,\,\vec{0}) \quad \text{and} \quad \vec{s}_*(\tau,\,\tau_0,\,\vec{x}_0,\,\vec{s}_0) \;=\; \left.\frac{\partial\vec{s}}{\partial\vec{s}_0}\right|_{\vec{s}_0=\vec{0}}\cdot\vec{s}_0$$

gives us the solution of (27).

b) If the system (26) admits an invariant function $V(\tau,\,\vec{z}\,)$ which can be represented in the form

$$V(\tau,\,\vec{z}\,) \;=\; V_m(\tau,\,\vec{z}\,) \;+\; V_{>m}(\tau,\,\vec{z}\,)$$

where V_m is a homogeneous polynomial of degree m in variables $\vec{s}$, and

$$\lim_{|\vec{s}|\to 0}\ \frac{V_{>m}}{|\vec{s}\,|^m} \;=\; 0$$

then $V_m(\tau,\,\vec{z}\,)$ is a first integral of the system (27).

c) If $\vec{x}(\tau,\,\tau_0,\,\vec{x}_0) \stackrel{\mathrm{def}}{=} \vec{\phi}(\tau,\,\tau_0,\,\vec{x}_0)$ is a solution of the first of the equations (27), then the system (27) can be written as a family of Hamiltonian systems of the type (26) depending on parameters $(\tau_0,\,\vec{x}_0)$ with the Hamiltonian function

$$\vec{W}\left(\tau,\,\vec{\phi}(\tau,\,\tau_0,\,\vec{x}_0)\right)\cdot\vec{s} \quad \text{where} \qquad \vec{W}(\tau,\,\vec{x}\,) \;=\; (\mathrm{grad}_{\vec{s}}\, H)\Big|_{\vec{s}=\vec{0}}$$

3 Hamiltonian Methods for the Extended System

3.1 Degenerate Poisson Brackets and Reduction of the Order of a Hamiltonian System

If there are nontrivial functions (**Casimir functions**) $f_l(\vec{z})$ (maybe given locally on the manifold M) such that

$$\{f_l,\ h\} \ = \ 0 \tag{28}$$

for any function $h(\vec{z})$ then the matrix $\hat{J}(\vec{z})$ is degenerate and this Poisson bracket is said to be **degenerate**. (For a degenerate matrix $\hat{J}(\vec{z})$ of constant rank, the functions f_l in (28) locally always exist.)

If all (at least all functionally independent) Casimir functions f_l have been found, then from properties (13) and (28) it follows that for an arbitrary Hamiltonian the trajectories of the system (11) $\vec{z}(\tau)$ lie on the intersecting level surfaces

$$f_l(\vec{z}) \ = \ c_l \ = \ const \quad (l \ = \ 1, 2, \ldots, m) \tag{29}$$

where the Poisson bracket no longer remains degenerate.

Locally any Poisson bracket of constant rank can be brought into the form (**Darboux theorem**)

$$\hat{J} \ = \ \begin{pmatrix} O_{nn} & I_n & O_{nm} \\ -I_n & O_{nn} & O_{nm} \\ O_{mn} & O_{mn} & O_{mm} \end{pmatrix} \tag{30}$$

where O_{kl} is a $k \times l$ zero matrix and I_n is a $n \times n$ identity matrix, and $\dim M \ = \ 2n + m$. Thus in local coordinates we obtain the classical Hamiltonian system with n degrees of freedom depending on m parameters ($c_1, c_2, \ldots, c_m$ in (29)).

3.2 Degenerate Poisson Brackets for Global Variables, or Local Darboux Coordinates ?

The spin-orbit Poisson bracket (18) is degenerate. It has the nontrivial Casimir function $f_1 = \mid \vec{s} \mid^2$ and on the level surface $f_1 = const > 0$ its rank

is constant and is equal to 8. This means that we can decrease the dimensions of the system (26) by introducing Darboux coordinates. Thus we obtain the **classical Hamiltonian system with 4 degrees of freedom depending on one parameter** $\mid \vec{s} \mid^2$. It is clear that the Darboux coordinates are not unique and may be introduced in various ways. We consider only one typical example.

Let $\vec{i},\ \vec{j},\ \vec{k}$ be an arbitrary orthogonal system of unit vectors in three dimensional space R^3 satisfying the condition

$$\vec{i} \cdot \left[\vec{j} \times \vec{k} \right] \ = \ 1$$

We introduce three new spin variables $\psi,\ J,\ I$ by the equations

$$\begin{cases} \vec{s} \cdot \vec{i} \ = \ J \\ \\ \vec{s} \cdot \vec{j} \ = \ \sqrt{I - J^2}\ \cos(\psi) \\ \\ \vec{s} \cdot \vec{k} \ = \ \sqrt{I - J^2}\ \sin(\psi) \end{cases} \tag{31}$$

or, equivalently

$$\vec{s}(\psi,\ J,\ I) \ = \ J \cdot \vec{i} \ + \ \sqrt{I - J^2}\ \left(\cos(\psi) \cdot \vec{j} \ + \ \sin(\psi) \cdot \vec{k} \right)$$

Here J is the projection of the spin vector on the $\vec{i}$-axis, ψ is the polar angle in the transverse plane and $I \ = \ \mid \vec{s} \mid^2$. In the new variables the spin part of motion equations (26) becomes

$$\dot{\psi} \ = \ H_J, \qquad \dot{J} \ = \ - H_\psi, \qquad \dot{I} \ = \ 0 \tag{32}$$

where the Hamiltonian (25) takes on the form

$$H(\tau,\ \vec{x},\ \psi,\ J,\ I) \ = \ H(\tau,\ \vec{x},\ \vec{s}(\psi,\ J,\ I))$$

Unfortunately, when

$$\left((\mathrm{grad}_{\vec{s}}\, H) \cdot \vec{j} \right)^2 \ + \ \left((\mathrm{grad}_{\vec{s}}\, H) \cdot \vec{k} \right)^2 \ \neq \ 0 \tag{33}$$

this coordinate system cannot be extended onto the whole sphere $I = const > 0$, since it has a singularity for $I - J^2 = 0$. This means we need to have a whole atlas of local coordinates systems (at least two local coordinate systems

defined by different vectors $\vec{i}_1,\ \vec{j}_1,\ \vec{k}_1$ and $\vec{i}_2,\ \vec{j}_2,\ \vec{k}_2$ in (31)) for a complete description of spin motion on the sphere in the electric and magnetic fields depending on time and position of the particle. We will have the same difficulties with any other Darboux coordinates because they are defined by the topological properties of the sphere. So the **way pointed by the Darboux theorem does not look like the most natural or straightforward approach to the problem of investigation of polarized beam dynamics** and we prefer to study the equations of spin-orbit motion using initial global variables $\vec{x},\ \vec{s}$ and the Poisson bracket (18) (degenerate).

Remark: Note that in Darboux coordinates considered here the Hamiltonian (20) linear with respect to spin variables takes on the form

$$H \ = \ H_{orbt} \ + \ \left(\vec{W}\cdot\vec{i}\right) J \ + \ \sqrt{I-J^2}\left(\left(\vec{W}\cdot\vec{j}\right)\cos\left(\psi\right) \ + \ \left(\vec{W}\cdot\vec{k}\right)\sin\left(\psi\right)\right)$$

and the condition (33) now reads as

$$\left(\vec{W}\cdot\vec{j}\right)^2 \ + \ \left(\vec{W}\cdot\vec{k}\right)^2 \ \neq \ 0$$

3.3 The Properties of Solutions of the Extended Equations of Spin-Orbit Motion Independent of the Specific Choice of the Hamiltonian

Which properties of solutions are independent of the specific choice of the Hamiltonian function in the equations of motion (11)? These are, for example, properties connected with the existence of Casimir functions and with the preservation of the Poisson bracket along the trajectories of the Hamiltonian system.

The spin-orbit Poisson bracket (18) has the Casimir function $\mid \vec{s} \mid^2$. This means that the length of the vector $\vec{s}(\tau)$ is preserved during the motion, i.e.

$$\mid \vec{s}(\tau) \mid \equiv \mid \vec{s}(\tau_0) \mid$$

Let $\vec{z}(\tau) \ = \ \vec{\phi}(\tau,\ \tau_0,\ \vec{z}_0)$ be the solution of the system (26), where $\vec{\phi}(\tau_0,\ \tau_0,\ \vec{z}_0) = \vec{z}_0$. Using the Taylor series expansion of the function $\vec{\phi}(\tau,\ \tau_0,\ \vec{z}_0)$ with respect to spin variables we obtain:

$$\begin{cases} \vec{x}(\tau) \ = \ \vec{F}(\tau,\ \tau_0,\ \vec{x}_0) \ + \ O(\mid \vec{s}_0 \mid) \\ \\ \vec{s}(\tau) \ = \ A(\tau,\ \tau_0,\ \vec{x}_0)\cdot\vec{s}_0 \ + \ O(\mid \vec{s}_0 \mid^2) \end{cases} \tag{34}$$

where $A(\tau,\, \tau_0,\, \vec{x}_0)$ is a 3×3 matrix. The map (34) preserves the Poisson bracket (18). Using this property we find:

a) The Jacobian matrix of the vector-function $\vec{F}(\tau,\, \tau_0,\, \vec{x}_0)$ is symplectic:

$$\left(\frac{\partial \vec{F}}{\partial \vec{x}_0}\right)^{\top} J \left(\frac{\partial \vec{F}}{\partial \vec{x}_0}\right) \;=\; J$$

Here the 6×6 matrix J is the symplectic unit [12] and the symbol '$\top$' indicates transpose of a matrix.

b) Every element of the matrix $A(\tau,\, \tau_0,\, \vec{x}_0)$ is equal to its own cofactor. For a 3×3 real nonsingular matrix this means that $A(\tau,\, \tau_0,\, \vec{x}_0)$ is an orthogonal matrix and $\det A(\tau,\, \tau_0,\, \vec{x}_0) \;=\; 1$, i.e.

$$A(\tau,\, \tau_0,\, \vec{x}_0) \;\in\; \mathrm{SO}(3)$$

We have briefly discussed the properties connected with the existence of Casimir functions and with the preservation of the Poisson bracket, but without doubt, Liouville's theorem on the conservation of volume is one of the most popular properties of classical Hamilton systems (at least in the accelerator physics) which is independent of the choice of a specific Hamiltonian. Strictly speaking, this is a statement about the existence of an integral invariant (dim $\vec{z}$-dimensional) of the density $g(\vec{z}) \;\equiv\; 1$.

It is well known [13] that the system of ordinary differential equations

$$\frac{d\vec{z}}{d\tau} \;=\; \vec{f}(\tau, \vec{z})$$

with continuously differentiable right hand side admits a non-negative continuously differentiable function $g(\vec{z})$ (in particular the function $g(\vec{z}) \equiv 1$) which serves as the density appearing in an integral invariant if and only if

$$\mathrm{div}_{\,\vec{z}}\, \left(g(\vec{z})\, \vec{f}(\tau,\, \vec{z})\right) \;=\; 0 \tag{35}$$

For the Hamiltonian system (12) the equality (35) has the form

$$\mathrm{div}_{\,\vec{z}}\, (g(\vec{z})\; \hat{J}(\vec{z})\; \mathrm{grad}_{\,\vec{z}}\, H(\tau, \vec{z})) \;=\; 0 \tag{36}$$

It follows from (36) that the Hamilton system (12) will preserve the phase space volume for a given Hamiltonian function H if and only if

$$\mathrm{div}_{\,\vec{z}}\, (\hat{J}(\vec{z})\; \mathrm{grad}_{\,\vec{z}}\, H) \;=\; \frac{\partial \hat{J}_{ij}}{\partial z_i} \cdot \frac{\partial H}{\partial z_j} \;=\; \sum_j \left(\sum_i \frac{\partial \hat{J}_{ij}}{\partial z_i}\right) \frac{\partial H}{\partial z_j} \;=\; 0 \tag{37}$$

Example: Defining in the two-dimensional Euclidean space the Poisson bracket by the equality

$$\{z_1,\ z_2\}\ =\ z_1$$

we will have the Hamiltonian system

$$\frac{dz_1}{d\tau}\ =\ z_1\,\frac{\partial H}{\partial z_2},\qquad \frac{dz_2}{d\tau}\ =\ -z_1\,\frac{\partial H}{\partial z_1} \tag{38}$$

Suppose $H\ =\ a\cdot z_2$. Then a solution of the system (38) is given by means of the formulae

$$z_2(\tau)\ =\ z_2(0),\qquad z_1(\tau)\ =\ z_1(0)\cdot e^{a\tau}$$

Consequently, an image of the single square

$$0\ \leq\ z_1(0)\ \leq\ 1,\qquad 0\ \leq\ z_2(0)\ \leq\ 1$$

will have the area $e^{a\tau}$ after a displacement along trajectories. The area is preserved for $a\ =\ 0$, and $\longrightarrow 0$ for $a\ <\ 0$, and $\longrightarrow\infty$ for $a>0$.

Examining the condition (37) we find that the Hamiltonian system will conserve phase space volume independently from the choice of Hamiltonian function if and only if

$$\left|\sum_i \frac{\partial \hat{J}_{ij}}{\partial z_i}\right|\ =\ 0,\qquad j\ =\ 1,\ldots,\dim\vec{z} \tag{39}$$

For the spin-orbit Poisson bracket all the values (39) vanish and consequently, the phase volume in the 9-dimensional space is preserved.

Moreover the phase volume is the integral invariant and in the reduced 8-dimensional phase space, this is a direct product of the two-dimensional sphere of a fixed radius $\rho>0$ and the 6-dimensional space of orbital variables (level surface of the Casimir function $\mid\vec{s}\mid^2$). To proof this fact consider an arbitrary sufficiently small domain D and introduce the spherical coordinates

$$\begin{cases} \vec{s}\cdot\vec{i}\ =\ \rho\cos\theta\sin\varphi \\ \\ \vec{s}\cdot\vec{j}\ =\ \rho\cos\theta\cos\varphi \\ \\ \vec{s}\cdot\vec{k}\ =\ \rho\sin\theta \end{cases} \tag{40}$$

where $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2},\ 0 \leq \varphi \leq 2\pi$ and where the unit vectors $\vec{i},\ \vec{j}$ and $\vec{k}$, satisfying the condition $\vec{i} \cdot \left[\vec{j} \times \vec{k}\right] = 1$ and forming an orthogonal basis, are chosen in such a way that the coordinate transformation (40) is nonsingular within some open set which includes the domain D. In new variables the equations of motion (26) have the form

$$\frac{dx}{d\tau} = J \operatorname{grad}_{\vec{x}} H, \qquad \frac{d\rho}{d\tau} = 0 \tag{41}$$

$$\frac{d\theta}{d\tau} = \frac{1}{\rho \cos\theta} \frac{\partial H}{\partial \varphi}, \qquad \frac{d\varphi}{d\tau} = -\frac{1}{\rho \cos\theta} \frac{\partial H}{\partial \theta} \tag{42}$$

If we neglect the equation

$$\frac{d\rho}{d\tau} = 0$$

it is easy to check that the remaining system of eight equations has the integral invariant with the density

$$g = \rho^2 \cos\theta \tag{43}$$

but the density (43) is just the density of a volume in the 8-dimensional phase space considered.

Remark: One can check that the properties listed above are correct not only for the extended but also for the original triangular system (in fact, this was the criterion for their selection). For the triangular system these properties can be regarded as properties independent of the choice of external electromagnetic field.

4 Canonical Transformations and the Introduction of Machine Coordinates for Circular Accelerators

In the theory of circular accelerators it is useful to describe the spin-orbit motion in terms of a curvilinear coordinate system associated with the design orbit. In the previous sections we have introduced Hamiltonian extension of the equations of classical spin-orbit motion. Hamiltonian systems have a very

special form, and the special form is not preserved by an arbitrary change of variables. In this section we describe the transformations of phase space that are canonical with respect to Poisson bracket (18) and preserve that special form, and which allow us to make coordinate transformations using the Hamiltonian function (20) directly instead of the equations of motion.

If we linearize the resulting Hamiltonian equations with respect to spin variables and then neglect the effect of spin on the orbit motion (**triangular truncation procedure**) we obtain the transformed version of the triangular system too.

4.1 Canonical Transformations of Phase Space

As was already noted, in a fixed local coordinate system $\vec{z}$ the Poisson bracket is completely defined if we know the values of the elements of the skew-symmetric matrix $\hat{J} = (\{z_i, z_j\})$ as functions of $\vec{z}$. Before introducing the canonical transformations we will discuss the converse problem. Under what conditions on the skew-symmetric $k \times k$ matrix $\tilde{J}(\vec{z})$ will the binary operation

$$F_{\tilde{J}}(f,\, g) \;=\; \mathrm{grad}_{\vec{z}}\, f \cdot \tilde{J}\, \mathrm{grad}_{\vec{z}}\, g$$

be a Poisson bracket? This operation is automatically bilinear and antisymmetric and satisfies the Leibnitz rule. Hence the operation $F_{\tilde{J}}(*,\, *)$ will be the Poisson bracket if and only if it satisfies the Jacobi identity. The Jacobi identity written in terms of elements of the matrix $\tilde{J}$ has the form (see, for example [14, 15])

$$\tilde{J}_{ml} \cdot \frac{\partial \tilde{J}_{ij}}{\partial y_l} \;+\; \tilde{J}_{jl} \cdot \frac{\partial \tilde{J}_{mi}}{\partial y_l} \;+\; \tilde{J}_{il} \cdot \frac{\partial \tilde{J}_{jm}}{\partial y_l} \;=\; 0, \quad i, j, m \;=\; 1, \ldots, k \tag{44}$$

Returning to the main theme, consider a coordinate transformation from old variables $\vec{z}$ to new variables $\vec{y}$ in the Hamiltonian system of ordinary differential equations (12)

$$\vec{z} \;=\; \vec{\varphi}(\vec{y}\,) \tag{45}$$

with a nondegenerate Jacobian matrix ($\dim \vec{z} \;=\; \dim \vec{y} \;=\; \dim M \;=\; k$)

$$\left(\frac{\partial \vec{\varphi}}{\partial \vec{y}}\right) \;=\; \begin{pmatrix} \frac{\partial \varphi_1}{\partial y_1} & \cdots & \frac{\partial \varphi_1}{\partial y_k} \\ \vdots & & \vdots \\ \frac{\partial \varphi_k}{\partial y_1} & \cdots & \frac{\partial \varphi_k}{\partial y_k} \end{pmatrix}$$

If we take into account the connection between the gradients of the Hamiltonian function H in old and new variables

$$\left(\frac{\partial\vec{\varphi}}{\partial\vec{y}}\right)^{-\top} \operatorname{grad}_{\vec{y}} H(\tau,\,\vec{\varphi}(\vec{y})) \;=\; \operatorname{grad}_{\vec{z}} H(\tau,\,\vec{z})\Big|_{\vec{z}=\vec{\varphi}(\vec{y})} \tag{46}$$

we easily obtain from (12) the differential equations for $\vec{y}$

$$\frac{d\vec{y}}{d\tau} \;=\; \left(\frac{\partial\vec{\varphi}}{\partial\vec{y}}\right)^{-1} \hat{J}(\vec{\varphi}(\vec{y}))\; \left(\frac{\partial\vec{\varphi}}{\partial\vec{y}}\right)^{-\top} \operatorname{grad}_{\vec{y}} H(\tau,\,\vec{\varphi}(\vec{y})) \tag{47}$$

We now introduce the function $\tilde{H}(\tau,\,\vec{y}\,) \;=\; H(\tau,\,\vec{\varphi}(\vec{y}))$ and the skew-symmetric matrix

$$\tilde{J}(\vec{y}\,) \;=\; \left(\frac{\partial\vec{\varphi}}{\partial\vec{y}}\right)^{-1} \hat{J}(\vec{\varphi}(\vec{y}))\left(\frac{\partial\vec{\varphi}}{\partial\vec{y}}\right)^{-\top}$$

and rewrite the system (47) in the form

$$\frac{d\vec{y}}{d\tau} \;=\; \tilde{J}(\vec{y}\,)\operatorname{grad}_{\vec{y}}\tilde{H} \tag{48}$$

Equation (48) resembles a Hamiltonian system with a new Hamiltonian function $\tilde{H}(\tau,\vec{y})$. But in fact it will actually only be a Hamiltonian system (independently from the choice of specific Hamiltonian) if the matrix $\tilde{J}$ satisfies the Jacobi identity (44). This condition looks very complicated, but there are two important situations when it becomes trivial:

1. The matrix $\tilde{J}(\vec{y}\,)$ is constant (independent from $\vec{y}\,$).

2. The matrix $\tilde{J}(\vec{y}\,)$ is equal to the matrix $\hat{J}(\vec{y}\,)$.

Example 1: For the Darboux coordinates (31) the matrix $\tilde{J}$ has the form (30) and hence is constant.

Example 2: Introduce the new spin-orbit variables by the equations

$$\begin{cases} \vec{x}_{old} \;=\; \vec{F}(\vec{x}_{new}) \\ \\ \vec{s}_{old} \;=\; \vec{s}_{new} \end{cases} \tag{49}$$

If the Jacobian matrix of the vector-function $\vec{F}$ is symplectic then the matrix $\tilde{J}$ is equal to the matrix $\hat{J}$ (24).

Now we are ready to introduce the necessary definitions.

Definition 1: *A coordinate transformation* $\vec{z} = \vec{\varphi}(\vec{y})$ *is a* **Poisson transformation** *of phase space if the matrix* $\tilde{J}(\vec{y})$ *satisfies the Jacobi identity (44).*

Definition 2: *A coordinate transformation* $\vec{z} = \vec{\varphi}(\vec{y})$ *is a* **Canonical transformation** *of phase space if the matrix* $\tilde{J}(\vec{y})$ *is equal to the matrix* $\hat{J}(\vec{y})$.

The Poisson transformations preserve the Hamiltonian form of the initial system, but their subset, canonical transformations, has additional helpful properties. Rewrite the condition $\tilde{J}(\vec{y}) = \hat{J}(\vec{y})$ in the form

$$\hat{J}(\vec{\varphi}(\vec{y})) = \left(\frac{\partial\vec{\varphi}}{\partial\vec{y}}\right) \hat{J}(\vec{y}) \left(\frac{\partial\vec{\varphi}}{\partial\vec{y}}\right)^{\top} \tag{50}$$

The equality (50) means that the image of the Poisson brackets of coordinates functions is equal to the Poisson brackets of the images. Thus a canonical transformation is an ″isometry″ of phase space, and we can find the Hamiltonian equations in new variables by just making the coordinate transformation in the Hamiltonian function (as in the usual way).

Example 3: The solution $\vec{z}(\tau) = \vec{\phi}(\tau, \tau_0, \vec{z}_0)$ of the Hamiltonian system (11) for any fixed $\tau \geq \tau_0$ is a canonical transformation of phase space:

$$\hat{J}(\vec{\phi}(\tau, \tau_0, \vec{z}_0)) = \left(\frac{\partial\vec{\phi}}{\partial\vec{z}_0}\right) \hat{J}(\vec{z}_0) \left(\frac{\partial\vec{\phi}}{\partial\vec{z}_0}\right)^{\top} \tag{51}$$

Remark 1: The condition for the map (45) to be symplectic is often used not in the form following from (50)

$$\left(\frac{\partial\vec{\varphi}}{\partial\vec{y}}\right) J \left(\frac{\partial\vec{\varphi}}{\partial\vec{y}}\right)^{\top} = J \tag{52}$$

where the matrix J is the symplectic unit, but in the form

$$\left(\frac{\partial\vec{\varphi}}{\partial\vec{y}}\right)^{\top} J \left(\frac{\partial\vec{\varphi}}{\partial\vec{y}}\right) = J \tag{53}$$

It can be easily shown that since $J^2 = -I$ these two conditions are equivalent, but it is not true in general that (52) and (53) are equivalent even for

a constant nondegenerate matrix $\hat{J}$ in (50). To illustrate this let us introduce a Poisson bracket in four-dimensional Euclidean space by means of the structural matrix

$$\hat{J} \;=\; \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & a \\ 0 & 0 & -a & 0 \end{pmatrix}, \qquad a \;=\; const$$

and consider the map

$$z_1 \;=\; y_1, \quad z_2 \;=\; y_2 - y_4, \quad z_3 \;=\; a\, y_1 + y_3, \quad z_4 \;=\; y_4$$

which is generated as a displacement along trajectories of the Hamiltonian system with $H = z_1 \cdot z_4$ for the time $\tau = 1$. Calculating the Jacobian matrix of this map we can check that (52) is satisfied but that (53) gives

$$\left(\frac{\partial \vec{z}}{\partial \vec{y}}\right)^{\top} \hat{J} \left(\frac{\partial \vec{z}}{\partial \vec{y}}\right) \;=\; \hat{J} + \begin{pmatrix} 0 & 0 & 0 & a^2-1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1-a^2 & 0 & 0 & 0 \end{pmatrix} \;\neq\; \hat{J} \quad \text{if} \quad a^2 \;\neq\; 1$$

Remark 2: Let a Poisson transformation satisfy the equality

$$\hat{J}(\vec{\varphi}(\vec{y})) \;=\; c \left(\frac{\partial \vec{\varphi}}{\partial \vec{y}}\right) \hat{J}(\vec{y}) \left(\frac{\partial \vec{\varphi}}{\partial \vec{y}}\right)^{\top} \tag{54}$$

for some nonzero constant c. If the relation (54) holds, then in the new local coordinates the structural matrix of the Poisson bracket is determined by the matrix $\tilde{J}(\vec{y}) = c\, \hat{J}(\vec{y})$. If $c \neq 1$ this transformation is not canonical. Sometimes for $c \neq 1$ it is useful to introduce for the transformed system a **new Poisson bracket**, determined by the matrix $\tilde{J}(\vec{y}) \;=\; \hat{J}(\vec{y})$. Defining the new Hamiltonian by means of the rule

$$\tilde{H}(\tau, \vec{y}) \;=\; c \cdot H(\tau, \vec{\varphi}(\vec{y}))$$

we can consider the transformed system as being Hamiltonian with the ″same″ Poisson bracket. For instance, in accelerator physics this method is used in studying betatron oscillations when the transverse momenta are normalized by the value of the kinetic momentum of a reference particle, which is assumed be a constant.

4.2 Nonautonomous Canonical Transformations

Except for transformations of the type (45), in this paper more general transformations of variables are used

$$\vec{z} \;=\; \vec{\varphi}(\tau,\, \vec{y}\,) \tag{55}$$

depending on τ as a parameter. Then we need to answer the question: ″In what case is the transformation (55) canonical and how is the new Hamiltonian to be calculated?″

Definition 3: *A coordinate transformation* $\vec{z} = \vec{\varphi}(\tau, \vec{y}\,)$ *is a* **nonautonomous canonical transformation** *if for any fixed* τ *the condition (50) holds and there is a differentiable function* $\hat{F}(\tau,\, \vec{y}\,)$ *such that*

$$\hat{J}(\vec{y}\,) \cdot grad_{\vec{y}}\; \hat{F} \;=\; -\left(\frac{\partial \vec{\varphi}}{\partial \vec{y}}\right)^{-1} \cdot \frac{\partial \vec{\varphi}}{\partial \tau} \tag{56}$$

Then the new Hamiltonian is given by the formula

$$\tilde{H}(\tau,\, \vec{y}\,) \;=\; H(\tau,\, \vec{\varphi}(\tau,\, \vec{y}\,)) \;+\; \hat{F}(\tau,\, \vec{y}\,)$$

Example 4: Defining in the two-dimensional Euclidean space the Poisson bracket by the equality

$$\{z_1,\; z_2\} \;=\; z_1 \;+\; z_2$$

it easy to check that the coordinate transformation

$$\begin{cases} z_1 \;=\; (1+b)\, y_1 \;+\; b\, y_2 \;+\; c \\ \\ z_2 \;=\; -b\, y_1 \;+\; (1-b)\, y_2 \;-\; c \end{cases} \tag{57}$$

is canonical independently from the choice of constants b and c. Letting now b and c be functions of τ we find that for the set $y_1 + y_2 \neq 0$ the equation (56) has the solution

$$\hat{F}(\tau,\, y_1,\, y_2) \;=\; -(y_1 + y_2)\, \frac{db}{d\tau} \;-\; \mathrm{sign}(y_1 + y_2)\, \ln|y_1 + y_2|\, \frac{dc}{d\tau}$$

and that this solution is unique up to an additive arbitrary function of the variable τ. This means that if

$$\frac{dc}{d\tau} \;\neq\; 0$$

we cannot consider the transformation (57) to be a nonautonomous canonical transformation because the equation (56) does not have differentiable solutions (at least in the neighbourhood of the set $y_1 + y_2 = 0$). So for general Poisson brackets the solvability of the equation (56) does not follow from satisfying the condition (50) for all values of τ.

We will not set ourselves the target of studying the general properties of nonautonomous canonical transformations, but instead we consider some examples directly connected with the purpose of this paper.

Example 5: Let $\vec{z}(\tau) = \vec{\phi}(\tau, \tau_0, \vec{z_0})$ be a solution of a canonical system with the Hamiltonian $\tilde{F}(\tau, \vec{z})$, so that

$$\frac{\partial \vec{\phi}}{\partial \tau} \;=\; \hat{J}(\vec{\phi}) \cdot \mathrm{grad}_{\vec{z}}\, \tilde{F}(\tau, \vec{z})\Big|_{\vec{z}=\vec{\phi}} \tag{58}$$

Taking into account (46) and (51) one obtains from (58)

$$\left(\frac{\partial \vec{\phi}}{\partial z_0}\right)^{-1} \frac{\partial \vec{\phi}}{\partial \tau} \;=\; \hat{J}(\vec{z}_0) \cdot \mathrm{grad}_{\vec{z}_0}\, \tilde{F}(\tau, \vec{\phi}(\tau, \tau_0, \vec{z}_0)) \tag{59}$$

From (59) and (51) it follows that a coordinate transformation

$$\vec{z} \;=\; \vec{\phi}(\tau, \tau_0, \vec{y})$$

is nonautonomous canonical and the new Hamiltonian is given by

$$\tilde{H}(\tau, \tau_0, \vec{y}) \;=\; H(\tau, \vec{\phi}(\tau, \tau_0, \vec{y})) \;-\; \tilde{F}(\tau, \vec{\phi}(\tau, \tau_0, \vec{y})) \tag{60}$$

As a particular case, the formula (60) contains the Hamiltonian version of the method of variation of constants when $H = H_1 + H_2$ and the function $\tilde{F}$ is chosen to be H_1 .

Example 6: Generalizing example 2 consider the transformation:

$$\begin{cases} \vec{x}_{old} \;=\; \vec{F}(\tau, \vec{x}_{new}) \\ \\ \vec{s}_{old} \;=\; \vec{s}_{new} \end{cases}$$

With the assumption that the Jacobian matrix of the vector function $\vec{F}(\tau, \vec{x})$ with respect to the variables $\vec{x}$ is symplectic for all values of τ , the equation (56) is reduced to

$$\mathrm{grad}_{\vec{x}}\, \hat{F} \;=\; \left(\frac{\partial \vec{F}}{\partial \vec{x}}\right)^{\top} J\, \frac{\partial \vec{F}}{\partial \tau} \tag{61}$$

The same symplecticity condition allows us to show that the matrix

$$\frac{\partial}{\partial \vec{x}} \left(\left(\frac{\partial \vec{F}}{\partial \vec{x}} \right)^{\top} J \, \frac{\partial \vec{F}}{\partial \tau} \right)$$

is symmetric and, consequently, that the equation (61) has a solution (defined up to some additive Casimir function).

Omitting a proof, we also point out that if only orbital variables are transformed, then we can use the classical technique of generating functions.

Example 7: The linear transformation of spin variables

$$\begin{cases} \vec{x}_{old} \;=\; \vec{x}_{new} \\ \\ \vec{s}_{old} \;=\; A(\tau)\, \vec{s}_{new} \end{cases} \tag{62}$$

will satisfy (50) if and only if $A \in \mathrm{SO}(3)$ for all values of τ. The condition (56) now becomes:

$$\begin{pmatrix} 0 & \frac{\partial \hat{F}}{\partial s_3} & -\frac{\partial \hat{F}}{\partial s_2} \\ \\ -\frac{\partial \hat{F}}{\partial s_3} & 0 & \frac{\partial \hat{F}}{\partial s_1} \\ \\ \frac{\partial \hat{F}}{\partial s_2} & -\frac{\partial \hat{F}}{\partial s_1} & 0 \end{pmatrix} \;=\; A^{\top} \, \frac{dA}{d\tau} \tag{63}$$

where the function $\hat{F}$ does not depend on $\vec{x}$.

Taking the derivative with respect to τ in the identity

$$A^{\top}(\tau)\, A(\tau) \;=\; I$$

we find that the matrix

$$A^{\top} \, \frac{dA}{d\tau}$$

is skewsymmetric, and hence that (63) has a solution which can be expressed as follows

$$\hat{F} \;=\; \left(A^{\top} \frac{dA}{d\tau} \right)_{23} \cdot s_1 \;-\; \left(A^{\top} \frac{dA}{d\tau} \right)_{13} \cdot s_2 \;+\; \left(A^{\top} \frac{dA}{d\tau} \right)_{12} \cdot s_3$$

So (62) will be a nonautonomous canonical transformation for an arbitrary differentiable matrix $A \in \mathrm{SO}(3)$.

4.3 The Coordinate Frame Connected with the Closed Design Orbit

In this section the words ″the closed design orbit″ mean some suitable closed curve which has a continuous unit tangent vector. Let the closed design orbit be described by the vector $\vec{r}_0(z)$, where z is the length along this curve. Supplement the unit tangent vector

$$\vec{T} \;=\; \frac{d\vec{r}_0}{dz}$$

with two unit vectors $\vec{N}$ and $\vec{B}$ satisfying the conditions

$$\vec{B} \;=\; \left[\vec{T}\times\vec{N}\right], \quad \vec{N} \;=\; \left[\vec{B}\times\vec{T}\right], \quad \vec{T} \;=\; \left[\vec{N}\times\vec{B}\right]$$

or, equivalently

$$\left[\vec{T}\times\vec{N}\right]\cdot\vec{B} \;=\; 1$$

The triplet $\vec{T}$, $\vec{N}$, $\vec{B}$ thus forms an orthogonal right handed coordinate system. We assume that the evolution of this coordinate system as the variable z changes is described by a periodic solution of the system of ordinary differential equations of the Fresnet type

$$\begin{aligned}
\frac{d\vec{T}}{dz} &= -h\vec{N} - \alpha\vec{B} \\
\frac{d\vec{N}}{dz} &= +h\vec{T} + æ\vec{B} \\
\frac{d\vec{B}}{dz} &= +\alpha\vec{T} - æ\vec{N}
\end{aligned}$$

We will find the transformation from the old spin-orbit coordinates to the new spin-orbit coordinates connected with the vectors $\vec{T}$, $\vec{N}$, $\vec{B}$ as a composition of two successive transformations: the first one changes the orbital variables and second one changes the spin variables and the longitudinal momentum.

4.3.1 Transformation of Orbital Variables

In the new coordinate system an arbitrary orbit-vector $\vec{r}$ lying in a sufficiently small neighbourhood of the closed design orbit can be written in the form

$$\vec{r} \;=\; \vec{r}_0(z) \;+\; x\,\vec{N} \;+\; y\,\vec{B}$$

We will take the parameters z, x, y to be the new orbital variables. The transition from the old coordinates to the new coordinates is made, as usual, with the help of the generating function depending on the new position and old momentum variables

$$F(\vec{r}, \vec{p}) \;=\; -\left(\vec{r}_0(z) \;+\; x\,\vec{N} \;+\; y\,\vec{B}\right)\cdot\vec{p}$$

The new momenta are given by the equations

$$P_z \;=\; -\frac{\partial F}{\partial z} \;=\; \vec{p}\cdot\left((1+hx+\alpha y)\,\vec{T} \;+\; æ\left(x\,\vec{B} \;-\; y\,\vec{N}\right)\right) \tag{64}$$

$$P_x \;=\; -\frac{\partial F}{\partial x} \;=\; \vec{p}\cdot\vec{N} \tag{65}$$

$$P_y \;=\; -\frac{\partial F}{\partial y} \;=\; \vec{p}\cdot\vec{B} \tag{66}$$

This transformation is a canonical transformation with respect to the spin-orbit Poisson bracket.

4.3.2 Transformation of Spin Variables and Longitudinal Momentum

The new spin variables are introduced via the equation

$$\vec{s}_{old} \;=\; C(z)\,\vec{s}_{new} \tag{67}$$

where $C(z) \;=\; \left(\vec{N}(z),\, \vec{B}(z),\, \vec{T}(z)\right)$ is a 3×3 matrix.

Since the matrix C depends on the variable z the coordinate transformation (67) is not canonical. To make it canonical we change the longitudinal momentum too:

$$P_z^{old} \;=\; P_z^{new} \;-\; \alpha\,s_x \;+\; h\,s_y \;-\; æ\,s_z \tag{68}$$

Here s_x, s_y, s_z are the components of the spin vector $\vec{s}_{new}$.

The coordinate transformation (67), (68) satisfies the condition (50) and hence is canonical. Since the old Hamiltonian contains all vectors as scalar and cross products then the new Hamiltonian will have the same form as the old one if we imagine that all vectors in (20) are written in terms of

the projections on the unit vectors $\vec{N},\ \vec{B},\ \vec{T}$ and we take into account the formulae

$$p_{\vec{N}} \ = \ \vec{p}\cdot\vec{N} \ = \ P_x, \qquad p_{\vec{B}} \ = \ \vec{p}\cdot\vec{B} \ = \ P_y$$

$$p_{\vec{T}} \ = \ \vec{p}\cdot\vec{T} \ = \ \frac{1}{1+hx+\alpha y}\,(P_z \ - \ æ\,(xP_y \ - \ yP_x) \ - \ \alpha\, s_x \ + \ h\, s_y \ - \ æ\, s_z)$$

which we can easily obtain from (64)-(66) and (68). Here we have reverted to using the symbol P_z instead of P_z^{new}.

4.4 Change of Independent Variable in Nonautonomous Hamiltonian Equations

Let the right hand part of the first equation of the Hamiltonian system (11) satisfy the condition

$$\frac{dz_1}{d\tau} \ = \ \{z_1,\ H(\tau,\ \vec{z})\} \ \neq \ 0 \tag{69}$$

This means that the variable $z_1(\tau)$ changes monotonically with changing τ (strictly increasing or strictly decreasing) so that one can introduce it as new independent variable. In many cases of practical importance the new ″time″ scale connected with z_1 gives us certain advantages and we wish to discuss the procedure of its introduction for Hamiltonian systems.

For convenience we introduce following notation

$$\vec{z} \ \stackrel{\text{def}}{=} \ (q,\ p,\ y_1,\ y_2,\ \ldots,\ y_{k-2}) \ = \ (q,\ p,\ \vec{y}) \tag{70}$$

for the components of the vector $\vec{z}$ and assume that the matrix $\hat{J}$ for the Hamiltonian equations (11) has the form

$$\hat{J}(\vec{z}) \ = \ \begin{pmatrix} 0 & 1 & 0 & \ldots & 0 \\ -1 & 0 & 0 & \ldots & 0 \\ 0 & 0 & & & \\ \vdots & \vdots & & \bar{J}(\vec{y}) & \\ 0 & 0 & & & \end{pmatrix} \tag{71}$$

This is not a restriction, because on the one hand the spin-orbit Poisson bracket (18) has the necessary form and on the other hand any matrix $\hat{J}$ can

be brought into this form if the condition (69) holds. With the new notations the system (11) becomes

$$\frac{dq}{d\tau} = \frac{\partial H}{\partial p}, \quad \frac{dp}{d\tau} = -\frac{\partial H}{\partial q}, \quad \frac{dy_i}{d\tau} = \bar{J}_{ij}(\vec{y})\frac{\partial H}{\partial y_j}, \quad i = 1, \ldots, k-2 \tag{72}$$

and the condition (69) now reads as

$$\frac{\partial H}{\partial p} \neq 0 \tag{73}$$

We will interpret the procedure of changing the independent variable as a procedure of the reduction of an autonomous Hamiltonian system to a family of nonautonomous Hamiltonian equations of smaller dimension defined on the level surfaces of the initial Hamiltonian function. With this aim in mind we introduce two additional canonical variables (E, υ) and the new Hamiltonian

$$\mathcal{H}(\upsilon, E, \vec{z}) = H(\upsilon, \vec{z}) - E$$

to obtain an autonomous Hamiltonian system in a higher dimensional phase space

$$\frac{dq}{d\tau} = \frac{\partial \mathcal{H}}{\partial p}, \qquad \frac{dp}{d\tau} = -\frac{\partial \mathcal{H}}{\partial q} \tag{74}$$

$$\frac{dy_i}{d\tau} = \bar{J}_{ij}(\vec{y})\,\frac{\partial \mathcal{H}}{\partial y_j}, \qquad i = 1, \ldots, k-2 \tag{75}$$

$$\frac{dE}{d\tau} = \frac{\partial \mathcal{H}}{\partial \upsilon}, \qquad \frac{d\upsilon}{d\tau} = -\frac{\partial \mathcal{H}}{\partial E} = 1 \tag{76}$$

Using the condition (73) we obtain the differential equations for the new independent variable q

$$\frac{dy_i}{dq} = \frac{dy_i}{d\tau}\cdot\frac{d\tau}{dq} = \bar{J}_{ij}(\vec{y})\,\frac{\partial \mathcal{H}/\partial y_j}{\partial \mathcal{H}/\partial p}, \quad i = 1, \ldots, k-2 \tag{77}$$

$$\frac{dE}{dq} = \frac{dE}{d\tau}\cdot\frac{d\tau}{dq} = \frac{\partial \mathcal{H}/\partial \upsilon}{\partial \mathcal{H}/\partial p}, \qquad \frac{d\upsilon}{dq} = \frac{d\upsilon}{d\tau}\cdot\frac{d\tau}{dq} = -\frac{\partial \mathcal{H}/\partial E}{\partial \mathcal{H}/\partial p} \tag{78}$$

We now wish to show that the equations (77), (78) are the family of nonautonomous Hamiltonian systems defined on the level surfaces

$$\mathcal{H}(E,\, \upsilon,\, q,\, p,\, \vec{y}) \;=\; c_0 \tag{79}$$

Let

$$p \;=\; K(E,\, \upsilon,\, \vec{y},\, q,\, c_0)$$

be the solution of the equation (79). According to the implicit function theorem this solution exists if the condition (73) holds. Taking the derivative with respect to the variable y_i in the identity

$$\mathcal{H}(E,\, \upsilon,\, q,\, K(E,\, \upsilon,\, \vec{y},\, q,\, c_0),\, \vec{y}) \;=\; c_0 \tag{80}$$

we have

$$0 \;=\; \frac{\partial \mathcal{H}}{\partial y_i} \;+\; \frac{\partial \mathcal{H}}{\partial p} \cdot \frac{\partial K}{\partial y_i} \tag{81}$$

From (81) it follows that

$$\frac{\partial K}{\partial y_i} \;=\; -\,\frac{\partial \mathcal{H}/\partial y_j}{\partial \mathcal{H}/\partial p} \tag{82}$$

Similarly we obtain

$$\frac{\partial K}{\partial E} \;=\; -\frac{\partial \mathcal{H}/\partial E}{\partial \mathcal{H}/\partial p}, \qquad \frac{\partial K}{\partial \upsilon} \;=\; -\frac{\partial \mathcal{H}/\partial \upsilon}{\partial \mathcal{H}/\partial p} \tag{83}$$

Comparing (77), (78) and (82), (83) and introducing the Hamiltonian function

$$\hat{H} \;=\; -\,K(E,\, \upsilon,\, \vec{y},\, q,\, c_0)$$

we see that the equations (77), (78) become the family of nonautonomous Hamiltonian systems depending on the parameter c_0

$$\frac{dE}{dq} = \frac{\partial \hat{H}}{\partial \upsilon}, \quad \frac{d\upsilon}{dq} = -\frac{\partial \hat{H}}{\partial E}, \quad \frac{dy_i}{dq} = \bar{J}_{ij}(\vec{y})\frac{\partial \hat{H}}{\partial y_j}, \quad i = 1, \ldots, k-2 \tag{84}$$

After solving the system (84) for a fixed value of c_0 we can find the dependence of p on q using the identity

$$p(q) \;=\; K(E(q),\, \upsilon(q),\, \vec{y}(q),\, q,\, c_0)$$

and then determine q as a function of τ from the equation

$$\tau \;-\; \tau_0 \;=\; \int\limits_{q_0}^{q} \frac{dq}{g(q)} \tag{85}$$

where

$$g(q) \;=\; \frac{\partial \mathcal{H}}{\partial p}(E(q),\, \upsilon(q),\, q,\, p(q),\, \vec{y}(q)) \;\neq\; 0$$

(equation (85) follows from the first of equations (72)).

Now we have to remember that we did not start from an autonomous system (74)-(76) but from nonautonomous system (72). This means that we do not need equation (85) because we have the dependence $\tau(q)$ from the second equation of (84) (υ is just another notation for τ). We also have freedom in the choice of the initial condition for the variable E. This means that we can choose a single fixed value of the parameter c_0 (usually $c_0 = 0$) and replace the initial nonautonomous Hamiltonian system by the nonautonomous Hamiltonian system (84) with the same matrix $\hat{J}$ as in (71).

4.5 Length Along the Design Orbit as Independent Variable

For a circular accelerator the spin-orbit Hamiltonian is always a periodic function of the variable z, but its dependence on time may be more complicated (for example, in the acceleration mode). This is one of the reasons for introducing the length along the design orbit as the independent variable. We will assume that

$$\frac{\partial H}{\partial P_z} \;\neq\; 0 \tag{86}$$

The condition (86) approximately means that during the motion in the accelerator the particle cannot reverse its direction.

Following the previous subsection we introduce two additional canonical variables $(E,\, \tau)$[5] and the new Hamiltonian

$$\mathcal{H} \;=\; H(\tau,\, x,\, P_x,\, y,\, P_y,\, z,\, P_z,\, s_x,\, s_y,\, s_z) \;-\; E$$

$$\frac{dE}{dt} \;=\; \frac{\partial \mathcal{H}}{\partial \tau}, \qquad \frac{d\tau}{dt} \;=\; -\,\frac{\partial \mathcal{H}}{\partial E} \;\equiv\; 1$$

Now we need to solve the equation

$$\mathcal{H}(E,\, \tau,\, x,\, P_x,\, y,\, P_y,\, z,\, P_z,\, s_x,\, s_y,\, s_z) \;=\; 0 \tag{87}$$

with respect to variable P_z to obtain the new Hamiltonian

$$\hat{H} \;=\; -\,P_z(E,\, \tau,\, x,\, P_x,\, y,\, P_y,\, s_x,\, s_y,\, s_z,\, z) \tag{88}$$

The dependence of the Hamiltonian $\mathcal{H}$ on the variable P_z is more complicated than in the pure orbital case, but nevertheless, we can solve the equation (87) with any required precision with respect to spin variables using the method of successive iterations.

4.6 The Hamiltonian in New Variables up to First Order with Respect to Spin Variables

In this subsection we discuss the general form of the spin-orbit Hamiltonian in the new variables $(E,\, t,\, x,\, P_x,\, y,\, P_y)$[6] up to the first order with respect to spin variables. This Hamiltonian is

$$\hat{H} \;=\; \hat{H}_{orbt}(E,\, t,\, x,\, P_x,\, y,\, P_y,\, z) \;+\; \hat{H}_{spin}(E,\, t,\, x,\, P_x,\, y,\, P_y,\, z,\, \vec{s}\,) \tag{89}$$

with

$$\hat{H}_{orbt} \;=\; -\,æ\,(x\,P_y \;-\; y\,P_x) \;-\; (1 \;+\; h\,x \;+\; \alpha\,y)\cdot$$

$$\cdot\left(\frac{e}{c}\,A_{\vec{T}} \;+\; \sqrt{\frac{(E \;-\; e\Phi)^2}{c^2} - m_0^2\,c^2 - \left(P_x - \frac{e}{c}\,A_{\vec{N}}\right)^2 - \left(P_y - \frac{e}{c}\,A_{\vec{B}}\right)^2}\right) \;=$$

[5]Reserving for a while the new symbol τ for the time t appearing in the spin-orbit Hamiltonian, and still reserving the symbol t for the independent variable.

[6]Here and in the Hamiltonian (88) we have reverted to using the symbol t instead of τ.

$$= \, - \, æ \, x \, \pi_{\vec{B}} \, + \, æ \, y \, \pi_{\vec{N}} \, - \, (1 \, + \, h \, x \, + \, \alpha \, y) \, \pi_{\vec{T}} \, - \, \frac{e}{c} \, A_z$$

and

$$\hat{H}_{spin} \, = \, - \, \alpha \, s_x \, + \, h \, s_y \, - \, æ \, s_z \, +$$

$$+ \, \frac{(1 \, + \, h \, x \, + \, \alpha \, y)(E \, - \, e \, \Phi)}{c^2 \sqrt{\frac{(E - e\Phi)^2}{c^2} \, - \, m_0^2 c^2 \, - \, \left(P_x - \frac{e}{c} A_{\vec{N}} \right)^2 - \left(P_y - \frac{e}{c} A_{\vec{B}} \right)^2}} \, \vec{W} \cdot \vec{s} \, =$$

$$= \, - \, \alpha \, s_x \, + \, h \, s_y \, - \, æ \, s_z \, + \, (1 \, + \, h \, x \, + \, \alpha \, y) \, \frac{m_0 \, \gamma}{\pi_{\vec{T}}} \, \vec{W} \cdot \vec{s}$$

Here $\vec{W}$ has the same form as in (20) with $\vec{\pi}$, $\vec{\mathcal{B}}$, $\vec{\mathcal{E}}$ written in terms of projections on the vectors $\vec{N}$, $\vec{B}$, $\vec{T}$

$$\pi_{\vec{N}} \, = \, P_x \, - \, \frac{e}{c} \, A_{\vec{N}}, \qquad \pi_{\vec{B}} \, = \, P_y \, - \, \frac{e}{c} \, A_{\vec{B}}$$

$$\pi_{\vec{T}} \, = \, \sqrt{\frac{(E - e\Phi)^2}{c^2} - m_0^2 c^2 - \left(P_x - \frac{e}{c} A_{\vec{N}} \right)^2 - \left(P_y - \frac{e}{c} A_{\vec{B}} \right)^2} \, =$$

$$= \, \left(\frac{(E - e\Phi)^2}{c^2} \, - \, m_0^2 c^2 \, - \, \pi_{\vec{N}}^2 \, - \, \pi_{\vec{B}}^2 \right)^{1/2}$$

and we use the notation

$$A_z \, = \, (1 \, + \, h \, x \, + \, \alpha \, y) \, A_{\vec{T}} \, + \, æ \, (x \, A_{\vec{B}} \, - \, y \, A_{\vec{N}})$$

The value of γ is defined now through the new canonical variable E

$$\gamma \, = \, \frac{E \, - \, e \, \Phi}{m_0 \, c^2}$$

To obtain the Hamiltonian (89) we have used the condition (86) in the form

$$\frac{\partial H}{\partial P_z} \, > \, 0$$

so that the vector $\vec{T}$ is chosen with the same orientation as the direction of particle flight in the accelerator.

To complete the description we also give the expressions for the projections on $\vec{N},\ \vec{B},\ \vec{T}$ of the electric and magnetic fields in terms of the vector and scalar potentials.

The magnetic field:

$$\mathcal{B}_{\vec{N}} \;=\; \frac{1}{1 \,+\, h\,x \,+\, \alpha\,y} \cdot \left(\frac{\partial A_z}{\partial y} \;-\; \frac{\partial A_{\vec{B}}}{\partial z} \;-\; æ\,y\,\mathcal{B}_{\vec{T}} \right)$$

$$\mathcal{B}_{\vec{B}} \;=\; \frac{1}{1 \,+\, h\,x \,+\, \alpha\,y} \cdot \left(\frac{\partial A_{\vec{N}}}{\partial z} \;-\; \frac{\partial A_z}{\partial x} \;+\; æ\,x\,\mathcal{B}_{\vec{T}} \right)$$

$$\mathcal{B}_{\vec{T}} \;=\; \frac{\partial A_{\vec{B}}}{\partial x} \;-\; \frac{\partial A_{\vec{N}}}{\partial y}$$

The electric field:

$$\mathcal{E}_{\vec{N}} \;=\; -\,\frac{\partial \Phi}{\partial x} \;-\; \frac{1}{c}\,\frac{\partial A_{\vec{N}}}{\partial t}$$

$$\mathcal{E}_{\vec{B}} \;=\; -\,\frac{\partial \Phi}{\partial y} \;-\; \frac{1}{c}\,\frac{\partial A_{\vec{B}}}{\partial t}$$

$$\mathcal{E}_{\vec{T}} \;=\; -\,\frac{1}{1 + hx + \alpha y} \left(\frac{\partial \Phi}{\partial z} \;+\; æ \left(y\,\frac{\partial \Phi}{\partial x} \;-\; x\,\frac{\partial \Phi}{\partial y} \right) \right) \;-\; \frac{1}{c}\,\frac{\partial A_{\vec{T}}}{\partial t}$$

(Useful formulae which allow us to get the equations for vector and scalar potentials in our curvilinear coordinate system may be found in the Appendix A).

5 Linear Differential Equations of Spin Motion

In this section we discuss the situation in which the Hamiltonian function is linear in the spin variables and does not depend on the orbit variables. In particular, this case includes the description of the behaviour of the spin vector on the closed (or any other chosen) trajectory of orbital motion, using the triangular system.

5.1 Matrix Representation of the Hamiltonian Function and Simple Properties of Solutions

The Hamiltonian function, which is linear in spin variables and does not depend on orbit variables, has the form

$$H(\tau,\, \vec{s}) \;=\; \vec{w}(\tau) \cdot \vec{s} \tag{90}$$

We now construct the skewsymmetric matrix $C(\vec{w})$ according to the rule

$$C(\vec{w}) \;=\; \begin{pmatrix} 0 & -w_3 & w_2 \\ w_3 & 0 & -w_1 \\ -w_2 & w_1 & 0 \end{pmatrix}$$

where w_1, w_2, w_3 are the components of the vector $\vec{w}$

$$\vec{w} \;=\; (w_1,\, w_2,\, w_3)$$

It is easy to verify that with the help of the matrix $C(\vec{w})$ the Hamiltonian (90) can be written as follows

$$H(\tau,\, \vec{s}) \;=\; \frac{1}{2}\, \vec{s} \cdot \operatorname{curl}_{\vec{s}}\, (C(\vec{w}) \cdot \vec{s}) \tag{91}$$

and the equations of motion take the form

$$\frac{d\vec{s}}{d\tau} \;=\; C\,(\vec{w}(\tau)) \cdot \vec{s} \tag{92}$$

The matrix representation (91) of the Hamiltonian function (90) resembles the representation of the Hamiltonian of linear orbit motion as a quadratic form and is particularly convenient for linear nonautonomous canonical transformations of variables. Introduce, for example, the new spin variables $\vec{u}$ by the equation

$$\vec{s} \;=\; A(\tau)\, \vec{u}, \qquad A(\tau) \;\in\; \mathrm{SO}(3) \tag{93}$$

Substituting (93) in the equation of motion (92), we obtain

$$\frac{d\vec{u}}{d\tau} \;=\; \left(A^{\top} C A \;-\; A^{\top} \frac{dA}{d\tau} \right) \cdot \vec{u}$$

where

$$A^{\top}C(\vec{w})A \;-\; A^{\top}\frac{dA}{d\tau}$$

is a skewsymmetric matrix again. Consequently, the Hamiltonian function in the new variables $\vec{u}$ can be written in the form

$$H(\tau,\,\vec{u}\,) \;=\; \frac{1}{2}\,\vec{u}\cdot\mathrm{curl}_{\,\vec{u}}\left(\left(A^{\top}CA \;-\; A^{\top}\frac{dA}{d\tau}\right)\cdot\vec{u}\right) \tag{94}$$

The following simple properties of the matrix notation are almost obvious:

a) $\mathrm{curl}_{\,\vec{s}}\;(\;A\cdot\vec{s}\,)\;+\;\mathrm{curl}_{\,\vec{s}}\;(\;B\cdot\vec{s}\,)\;=\;\mathrm{curl}_{\,\vec{s}}\;(\;(A+B)\cdot\vec{s}\,)$

b) $A^{\top}C(\vec{w})\,A\;=\;C\left(A^{\top}\cdot\vec{w}\right)\quad$ for $\quad A\;\in\;\mathrm{SO}(3)$

Let $M(\tau,\,\tau_0)$ be the fundamental matrix solution of (92). It has been mentioned in subsection 3.3 that for any $\tau\;\geq\;\tau_0$

$$M(\tau,\,\tau_0)\;\in\;\mathrm{SO}(3) \tag{95}$$

If the vector $\vec{w}$ is constant (does not depend on τ) and $|\vec{w}|\;\neq\;0$, then the matrix M is determined via the matrix $C(\vec{w})$ from the formula

$$M(\tau,\,\tau_0)\;=\;\exp\left((\tau-\tau_0)\,C(\vec{w})\right)\;=$$

$$I+\sin(\mid\vec{w}\mid(\tau-\tau_0))\cdot\frac{C(\vec{w})}{\mid\vec{w}\mid}+(1-\cos(\mid\vec{w}\mid(\tau-\tau_0)))\cdot\left(\frac{C(\vec{w})}{\mid\vec{w}\mid}\right)^{2} \tag{96}$$

In the case $|\vec{w}|\;=\;0$ the matrix M is equal to the identity matrix.

The properties of the solutions of the system of linear differential equations (92) given below follow from the Hamiltonian character of the system (92) (in the case studied here it is equivalent to the fact that the matrix $C(\vec{w})$ in (92) is real skewsymmetric):

a) The norm of every solution is conserved.

b) The angle between any two solutions is conserved.

c) If $\vec{n}(\tau)$ and $\vec{m}(\tau)$ are solutions of the system (92), then $\vec{l}(\tau)\;=\;[\vec{n}(\tau)\times\vec{m}(\tau)]$ is a solution too.

d) The function $\vec{n}(\tau) \cdot \vec{s}$ is a constant of motion of the system (92) if and only if $\vec{n}(\tau)$ is a solution.

e) If the vector $\vec{w}(\tau)$ in (90) has period T in τ, then the system (92) has a T-periodic solution.

f) If the vector $\vec{w}(\tau)$ in (90) has period T in τ and the system (92) has two linearly independent T-periodic solutions, then any solution of the system (92) is T-periodic.

5.2 Linear Equations of Spin Motion with Periodic Coefficients

Now we consider a Hamiltonian function (90) in which the vector $\vec{w}$ has period 2π in τ, that is

$$\vec{w}(\tau + 2\pi) \equiv \vec{w}(\tau)$$

We shall attempt to obtain a Hamiltonian function (94) of the simplest form by means of the linear coordinate substitution (93) with a 2π-periodic matrix $A(\tau)$.

5.2.1 Some Properties of SO(3) Matrices

Recall that the symbol SO(3) denotes the group of 3×3 real orthogonal matrices with the determinant equal to 1.

Let $A \in \mathrm{SO}(3)$. Since $A^{-1} = A^{\top}$, the spectrum of this matrix is symmetric about the unit circle (that is, if μ is an eigenvalue of a SO(3) matrix, then so is μ^{-1}). From the fact that the matrix is real it follows that the spectrum is also symmetric about the real axes. Taking into account the condition $\det A = 1$, we obtain that the eigenvalues of the matrix A have the form

$$\mu_1 = 1, \qquad \mu_{2,3} = \cos\lambda \pm i \sin\lambda$$

where

$$\cos\lambda = \frac{1}{2}\left(\mathrm{Tr}(A) - 1\right)$$

The next three lemmas give us some additional helpful properties of SO(3) matrices.

Lemma 1: *A 3×3 nonsingular real matrix A belongs to the* SO(3) *group if and only if every element of the matrix A is equal to its own cofactor.*

Note, if every element of a 3×3 real singular matrix A is equal to its own cofactor then $A = 0$.

Lemma 2: *All the eigenvalues of a matrix $A \in$ SO(3) are distinct if and only if $A \neq A^\top$ (or, equivalently, $A^2 \neq I$).*

Lemma 3: *If all the eigenvalues of a matrix $A \in$ SO(3) are distinct then the nonzero vector*

$$\vec{n} = \frac{\vec{k}}{|\vec{k}|}, \quad \vec{k} = (a_{32} - a_{23}, \quad a_{13} - a_{31}, \quad a_{21} - a_{12})$$

is a unit eigenvector of the matrix A corresponding to the unit eigenvalue, i.e. $A\vec{n} = \vec{n}$.

5.2.2 Real Jordan Canonical Forms of SO(3) Matrices

Let $\vec{n}$ be an eigenvector of A corresponding to the unit eigenvalue

$$A\,\vec{n} \;=\; \vec{n}, \qquad |\,\vec{n}\,| = \;1$$

Supplement the vector $\vec{n}$ with two unit vectors $\vec{m}$ and $\vec{l}$ satisfying the condition

$$\vec{m}\cdot\left[\vec{l}\times\vec{n}\right] \;=\; 1$$

to form an orthogonal basis. Then the matrix

$$B \;=\; \left(\vec{m},\,\vec{l},\,\vec{n}\right) \;\in\; \mathrm{SO}(3)$$

In the new basis, constructed with vectors $\vec{m}$, $\vec{l}$ and $\vec{n}$, the matrix A will have the form

$$\bar{A} \;=\; B^{-1}AB \;=\; B^{\top}AB \;=\; \begin{pmatrix} A\vec{m}\cdot\vec{m} & A\vec{l}\cdot\vec{m} & 0 \\ A\vec{m}\cdot\vec{l} & A\vec{l}\cdot\vec{l} & 0 \\ 0 & 0 & 1 \end{pmatrix} \;\in\; \mathrm{SO}(3)$$

It follows from the condition $\bar{A} \in$ SO(3) that for some angle ψ we can represent this matrix in the form

$$\bar{A} \;=\; \begin{pmatrix} +\cos(\psi) & +\sin(\psi) & 0 \\ -\sin(\psi) & +\cos(\psi) & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Using the equality $AB = B\bar{A}$, we obtain

$$A \cdot \vec{m} = \cos(\psi) \cdot \vec{m} - \sin(\psi) \cdot \vec{l}$$

$$A \cdot \vec{l} = \sin(\psi) \cdot \vec{m} + \cos(\psi) \cdot \vec{l}$$

or, in complex form,

$$A \cdot \left(\vec{m} + i\,\vec{l} \right) = \exp(i\,\psi) \cdot \left(\vec{m} + i\,\vec{l} \right) \tag{97}$$

It follows from (97) that $\exp(\pm\, i\,\psi)$ are the eigenvalues and $\vec{m} \pm i\,\vec{l}$ are the corresponding eigenvectors of the matrix A (in particular, $\psi = \pm\lambda \pmod{2\pi}$)

The matrix $\bar{A}$ is a **real Jordan canonical form** of the matrix A. We will denote it by $\bar{A}_1$. All possible canonical forms are given by the list

$$\bar{A}_1 = \begin{pmatrix} +\cos(\psi) & +\sin(\psi) & 0 \\ -\sin(\psi) & +\cos(\psi) & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \bar{A}_2 = \begin{pmatrix} +\cos(\psi) & -\sin(\psi) & 0 \\ +\sin(\psi) & +\cos(\psi) & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\bar{A}_3 = \begin{pmatrix} +\cos(\psi) & 0 & +\sin(\psi) \\ 0 & 1 & 0 \\ -\sin(\psi) & 0 & +\cos(\psi) \end{pmatrix}, \quad \bar{A}_4 = \begin{pmatrix} +\cos(\psi) & 0 & -\sin(\psi) \\ 0 & 1 & 0 \\ +\sin(\psi) & 0 & +\cos(\psi) \end{pmatrix}$$

$$\bar{A}_5 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & +\cos(\psi) & +\sin(\psi) \\ 0 & -\sin(\psi) & +\cos(\psi) \end{pmatrix}, \quad \bar{A}_6 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & +\cos(\psi) & -\sin(\psi) \\ 0 & +\sin(\psi) & +\cos(\psi) \end{pmatrix}$$

It is clear that all matrices $\bar{A}_i$ are similar one to another. For example,

$$B_i^{\top}\, \bar{A}\, B_i = \bar{A}_i, \qquad i = 1, \ldots, 6$$

where $B_i \in \mathrm{SO}(3)$ and

$$B_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad B_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad B_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$$

$$B_4 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad B_5 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad B_6 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix}$$

We note that if $\psi = 0 \pmod{2\pi}$, then all matrices $\bar{A}_i$ are equal one to another (and equal to the identity matrix). If $\psi = \pi \pmod{2\pi}$, then $\bar{A}_1 = \bar{A}_2$, $\bar{A}_3 = \bar{A}_4$ and $\bar{A}_5 = \bar{A}_6$.

5.2.3 Skew-symmetric Real Logarithm of SO(3) Matrices

If the determinant of the matrix A is nonzero, then

$$\ln(A) \;=\; K$$

is defined as a solution of the equation

$$\exp(K) \;=\; A \tag{98}$$

We now consider the problem of finding skewsymmetric real solutions of equation (98) in the case, when $A \in \mathrm{SO}(3)$.

Lemma 4: *If $A \in \mathrm{SO}(3)$ and $A \neq A^{\top}$, then all real logarithms of the matrix A are skewsymmetric matrices and can be expressed by formula*

$$\ln(A) \;=\; \frac{2\,\ln\lambda}{\lambda \,-\, \lambda^*}\left(A \,-\, A^{\top}\right)$$

where λ and λ^* are complex conjugate eigenvalues of the matrix A distinct from unity.

If $A = A^{\top}$ then not all real solutions of equation (98) are skewsymmetric matrices. For example

$$\exp\begin{pmatrix} 0 & wd & 0 \\ -w/d & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \;=\; \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \;\in\; \mathrm{SO}(3)$$

for all real $d \neq 0$ and all real $w = \pi \,(\mathrm{mod}\, 2\pi)$.

Lemma 5: *Let $\vec{n}$, $\vec{m}$, and $\vec{l}$ be the orthogonal basis connected with matrix $A \in \mathrm{SO}(3)$ and defined in the previous subsubsection. Then all real skewsymmetric logarithms of the matrix A are given by the formula*

$$\ln(A) \;=\; (\omega \,+\, 2\pi k)\cdot\begin{pmatrix} 0 & -n_3 & n_2 \\ n_3 & 0 & -n_1 \\ -n_2 & n_1 & 0 \end{pmatrix} \;=\; (\omega \,+\, 2\pi k)\cdot C(\vec{n})$$

where real ω $(0 \,\leq\, \omega \,<\, 2\,\pi)$ satisfies the equation [7]

$$\exp(i\omega) \;=\; \frac{1}{2}\left(\vec{m} \,+\, i\,\vec{l}\right)\cdot A\left(\vec{m} \,+\, i\,\vec{l}\right) \tag{99}$$

[7] Here and further on for complex vectors $\vec{u}$, $\vec{v} \in C^n$ we define $\vec{u}\cdot\vec{v} = u_1\cdot v_1^* + \ldots + u_n\cdot v_n^*$.

and k *is an arbitrary whole number.*

Note that the right hand side in (99) is just another expression for one of the eigenvalues (which we say is "conjugate with vector $\vec{n}$") of the matrix A, and hence the equation (99) always has a real solution.

Below in this paper we will use the notation $\ln_s A$ for real skewsymmetric values of the function $\ln A$.

Remark: In accordance with Lemma 5

$$\ln_s I \;=\; 2\,\pi\,k\cdot C(\vec{n}\,)$$

where $\vec{n}$ and k are an arbitrary unit vector and an arbitrary integer respectively.

5.2.4 Normal Forms for Hamiltonians of Linear Equations of Spin Motion with Periodic Coefficients

For simplicity fix $\tau_0 \;=\; 0$ and let $M(\tau)$ be the fundamental matrix of solution of the system (92). Then

$$M(\tau+2\pi) \;=\; M(\tau)\cdot M(2\pi)$$

Definition 1: *A transformation of variables*

$$\vec{s} \;=\; A(\tau)\vec{u}, \qquad A(\tau) \;\in\; \mathrm{SO}(3)$$

is called a **normalizing transformation**, *if*

a) $A(\tau+2\pi) \;\equiv\; A(\tau)$

b) *the Hamiltonian in the variables* $\vec{u}$ *does not depend on* τ

c) *the matrix* $A^{\top}(2\pi)\cdot M(2\pi)\cdot A(2\pi)$ *is a real Jordan canonical form*

Definition 2: *The result of the application of the normalizing transformation to the Hamiltonian (90) with the periodic vector* $\vec{w}(\tau)$ *is called the* **normal form** *of the Hamiltonian for linear equations of spin motion with periodic coefficients.*

Let $\bar{M}$ be one of the real Jordan canonical forms of the matrix $M(2\pi)$ and let the matrix $B \in \mathrm{SO}(3)$ be such that

$$B^{\top} M(2\pi)\, B \;=\; \bar{M}$$

Lemma 6: *The matrix*

$$A(\tau) \;=\; M(\tau)\exp\left(-\frac{\tau}{2\pi}\ln_s M(2\pi)\right)B$$

defines the normalizing transformation, which takes the initial Hamiltonian (90) into the normal form

$$\bar{H}(\vec{s}) \;=\; \frac{1}{2}\,\vec{s}\cdot\mathrm{curl}_{\vec{s}}\left(\frac{1}{2\pi}\left(\ln_s \bar{M}\right)\cdot\vec{s}\right) \tag{100}$$

Examining all possible real Jordan canonical forms of the matrix $M(2\pi)$ and explicitly calculating the logarithm in (100) we get

Lemma 7: *The Hamiltonian (90) with the 2π-periodic vector $\vec{w}(\tau)$ can be reduced to the following normal forms*

$$\bar{H}(\vec{s}) \;=\; (\pm\lambda \;+\; k)\cdot s_m \tag{101}$$

where

$$\lambda \;=\; \frac{1}{2\pi}\,\arccos\left(\frac{1}{2}(Tr(M(2\pi))-1)\right)$$

and the sign $'+'$ or $'-'$ in front of λ, $m\in\{1,2,3\}$ and the integer k can be chosen arbitrarily.

The number $\pm\lambda+k$ in (101) is called the **spin tune**. Using the freedom of the choice of k and of the sign $\pm$ in front of λ we can normalize the spin tune to a value lying in the region from 0 to 0.5.

Note, that if $A\neq I$ then the set of Hamiltonians (101) contains all possible normal forms. This is easy to see from the fact that if the Hamiltonian

$$\frac{1}{2}\,\vec{s}\cdot\mathrm{curl}_{\vec{s}}(C\cdot\vec{s})$$

is a normal form, then the matrix $\exp(2\pi C)$ is a real Jordan canonical form of the matrix $M(2\pi)$.

Another way to bring a 2π-periodic Hamiltonian linear in spin to the simplest form with help of a 2π-periodic coordinate transformation can be extracted from the results of Appendix B.

5.3 Connection between the SO(3) and SU(2) Groups and the Lax Form of the Equations of Spin Motion

The fundamental matrix solution $M(\tau,\, \tau_0)$ of the linear differential equations of spin motion (92) is a 3×3 matrix and consists of 9 elements, but since $M(\tau,\, \tau_0) \in \mathrm{SO}(3)$ it can be described completely with the help of a smaller number of parameters using the connection between the SO(3) and SU(2) groups. Usually in accelerator physics this connection is established in the framework of the spinor formalism [16, 17]. The method we shall use in this subsection (following [6]) gives us the same results, but does not use the concept of spinors (at least in explicit form).

Let us recall the definition of the SU(2) group:

a 2×2 matrix U with complex coefficients belongs to the SU(2) group if $\det U = 1$ and $U \cdot U^* = I$ (the asterisk '$*$' indicates complex conjugation of a matrix).

From this definition it follows that any matrix $U \in \mathrm{SU}(2)$ has the form

$$U \;=\; \begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix}, \qquad a \cdot a^* \,+\, b \cdot b^* \;=\; 1 \tag{102}$$

We now define the matrix

$$L \;=\; \begin{pmatrix} s_3 & s_1 + i s_2 \\ s_1 - i s_2 & -s_3 \end{pmatrix}, \qquad L^* \;=\; L \tag{103}$$

corresponding to the vector $\vec{s}$ and introduce the anti-Hermitian matrix B

$$B \;=\; \frac{i}{2} \cdot \begin{pmatrix} w_3 & w_1 + i w_2 \\ w_1 - i w_2 & -w_3 \end{pmatrix}, \qquad B^* \;=\; -B \tag{104}$$

By means of the matrices L and B one can write the equations of spin motion (92) in the form of a **Lax equation**

$$\frac{dL}{d\tau} \;=\; B \cdot L \;-\; L \cdot B \tag{105}$$

Note that the right hand side of equation (105) satisfies the condition

$$(B \cdot L \;-\; L \cdot B)^* \;=\; B \cdot L \;-\; L \cdot B$$

Lemma 8: *If the matrix $U(\tau,\, \tau_0)$ satisfies the equation*

$$\frac{dU}{d\tau} \;=\; B \cdot U, \qquad U(\tau_0,\, \tau_0) \;=\; I \quad (\mathit{or}\;\; U(\tau_0,\, \tau_0) \;=\; -I) \tag{106}$$

then $U(\tau,\,\tau_0)\,\in\,\mathrm{SU}(2)$ *and the solution of (105) is given by the formula*

$$L(\tau,\,\tau_0)\;=\;U(\tau,\,\tau_0)\cdot L(\tau_0,\,\tau_0)\cdot U^*(\tau,\,\tau_0) \tag{107}$$

Remark 1: Because (107) is a similarity transformation it follows that for any $\tau\,\geq\,\tau_0$ the matrices $L(\tau,\,\tau_0)$ and $L(\tau_0,\,\tau_0)$ have the same eigenvalues. In our case this means that for any $\tau\,\geq\,\tau_0$

$$\mid\vec{s}(\tau)\mid\equiv\mid\vec{s}(\tau_0)\mid$$

Remark 2: The equation of spin motion (92) can be written in the form (105) using another choice of the matrices L and B, different from (103) and (104). See, for example, [6].

The following formulae give us the connection between the matrix $U(\tau,\,\tau_0)$ in (107) and the fundamental matrix solution of (92) $M(\tau,\,\tau_0)$:

$$M\;=\;\begin{pmatrix} Re(a^2-b^2) & -Im(a^2+b^2) & -2\cdot Re(ab) \\ Im(a^2-b^2) & Re(a^2+b^2) & -2\cdot Im(ab) \\ 2\cdot Re(ab^*) & -2\cdot Im(ab^*) & aa^*-bb^* \end{pmatrix} \tag{108}$$

Introduce the quantities

$$v\;=\;\frac{1}{2}(m_{11}\,+\,m_{22}\,+\,i\,(m_{21}\,-\,m_{12}))\;=\;a^2$$

$$w\;=\;\frac{1}{2}(m_{22}\,-\,m_{11}\,-\,i\,(m_{21}\,+\,m_{12}))\;=\;b^2$$

where m_{kl} are the elements of the matrix M. Since for the orthogonal matrix M we have

$$w\cdot w^*\,+\,v\cdot v^*\;=\;\frac{1}{2}(m_{11}^2\,+\,m_{12}^2\,+\,m_{21}^2\,+\,m_{22}^2)\;\neq\;0$$

then using (102) and (108) we can define the elements of the matrix U by the equations

$$\begin{cases} a\;=\;\pm\sqrt{v}, \quad b\;=\;-\frac{1}{2a}(m_{13}\,+\,i\,m_{23}), \quad \text{if } v\,\neq\,0, \\ b\;=\;\pm\sqrt{w}, \quad a\;=\;-\frac{1}{2b}(m_{13}\,+\,i\,m_{23}), \quad \text{otherwise.} \end{cases}$$

Remark 3: The freedom of the choice of sign in these formulae (and also in (106)) is connected with the fact that the SU(2) group overlaps the SO(3) group twice.

If the vector $\vec{w}$ does not depend on τ, then the solution of equation (106) has the form

$$U(\tau,\,\tau_0) \;=\; U(\tau_0,\,\tau_0)\left(\cos\left(\frac{|\vec{w}|}{2}(\tau-\tau_0)\right)\cdot I \;+\; \frac{2}{|\vec{w}|}\sin\left(\frac{|\vec{w}|}{2}(\tau-\tau_0)\right)\cdot B\right)$$

where $U(\tau_0,\,\tau_0) \;=\; \pm I$ and we assumed $|\vec{w}| \;\neq\; 0$. In the case $|\vec{w}| \;=\; 0$ the matrix $U(\tau,\,\tau_0) \;\equiv\; U(\tau_0,\,\tau_0)$.

The usage of the SU(2) representation of SO(3) matrices not only allows us to reduce the number of free parameters, but is also helpful for some analytical calculations. Consider, for example, the matrix

$$\hat{L} \;=\; i\,(Re(a)\cdot I \;-\; U) \;=\; \begin{pmatrix} Im(a) & Im(b) - i\cdot Re(b) \\ Im(b) + i\cdot Re(b) & -Im(a) \end{pmatrix}$$

It is easy to verify that this matrix commutes with the matrix U, i.e.

$$\hat{L} \;=\; U\,\hat{L}\,U^* \tag{109}$$

Comparing (109) and (107) we get

Lemma 9: *Let U be a* SU(2) *matrix, corresponding to a given $A \in$* SO(3)*. If $A \neq I$ then*

$$\vec{n} \;=\; \frac{1}{\sqrt{1-(Re(a))^2}}\;(Im(b),\,-Re(b),\,Im(a))$$

is a unit eigenvector of the matrix A, corresponding to unit eigenvalue.

6 Normal Forms for the Spin-Orbit Hamiltonian

In this section we will consider the Hamiltonian system associated with the coupled spin-orbit Poisson bracket in a neighbourhood of a stationary point (or periodic solution) which can be canonically transformed to be the

origin[8]. In order to understand the properties of the solutions of such systems it is helpful firstly to find a coordinate substitution which reduces the original equations to the simplest possible form. Here we describe an algorithm which allows us to make coordinate transformations working not with equations of motion, but directly with the Hamiltonian function.

6.1 Canonical Transformations of Spin-Orbit Variables which Map the Origin into Itself

We shall be working with a phase space consisting of $2n + 3$ variables

$$\vec{z} \;=\; (\vec{x},\, \vec{s}) \;=\; (\vec{q},\, \vec{p},\, \vec{s}) \;=\; (q_1,\, \ldots,\, q_n,\, p_1,\, \ldots,\, p_n,\, s_1,\, s_2,\, s_3)$$

In this subsection we shall study canonical (i.e. the Poisson bracket (18) preserving) nonsingular (i.e. with nondegenerate Jacobian matrix) maps

$$\vec{z}_f \;=\; \vec{Z}(\vec{z}_i) \tag{110}$$

where the components of the real vector function $\vec{Z}(\vec{z}\,)$ are power series in $\vec{z}$ without constant terms.

6.1.1 Linear Canonical Transformations

The map which linearizes any symplectic map in the neighborhood of a fixed point is always a linear symplectic map. Does this apply to our case, where we are dealing with maps which are not symplectic but instead are canonical with respect to the Poisson bracket (18)?

Lemma 1: *The nonsingular real linear transformation*

$$\vec{z}_f \;=\; A\,\vec{z}_i$$

is canonical if and only if the matrix A has the form

$$A \;=\; \mathrm{diag}\,(A_{orbt},\, A_{spin})$$

[8] If $\vec{z}_*(\tau)$ is a solution of a Hamiltonian system associated with the coupled spin-orbit Poisson bracket then the parallel displacement $\vec{z}_{new} = \vec{z}_{old} - \vec{z}_*(\tau)$ will be a canonical transformation if and only if $\vec{s}_*(\tau) \equiv \vec{0}$. The question of what type of simple canonical coordinates can be introduced for an arbitrary solution is studied in Appendix B.

where

$$A_{orbt} \in \mathrm{Sp}(2n), \qquad A_{spin} \in \mathrm{SO}(3)$$

In more detail the map (110) can be written as

$$\begin{cases} \vec{x}_f = \vec{X}(\vec{x}_i, \vec{s}_i) \\ \\ \vec{s}_f = \vec{S}(\vec{x}_i, \vec{s}_i) \end{cases}$$

Using the condition of spin-orbit Poisson bracket preservation (50) it is possible to show that always

$$\vec{S}\left(\vec{x}, \vec{0}\right) = \vec{0}$$

Thus we have the following:

Lemma 2: *The linearization of the canonical map (110) is a linear canonical map if and only if*

$$\frac{\partial \vec{X}}{\partial \vec{s}} \left(\vec{0}, \vec{0}\right) = 0$$

So the answer to the above question is negative in the general case. The linearization of the nonsingular canonical map (in its usual meaning) is not bound to be a linear canonical transformation.

Definition 1: *A polynomial* $P(\vec{z}) = P(\vec{x}, \vec{s})$ *is* **quasi-homogeneous** *of degree* m *in its arguments* $\vec{x}$, $\vec{s}$ *if it satisfies the equation*

$$P(t\,\vec{x}, t^2\,\vec{s}) = t^m \cdot P(\vec{x}, \vec{s})$$

for every value of t.

The set of all quasi-homogeneous polynomials of degree m ($m = 0, 1, 2, \ldots$) will be denoted by $\mathcal{H}_s(m)$.

Definition 2: *The* **quasi-linearization** *of the map (110) is defined as a linear transformation*

$$\vec{z}_f = A_s\,\vec{z}_i$$

which acts invariantly on classes $\mathcal{H}_s(m)$ [9] *and minimizes the Euclidean norm of the difference*

$$\frac{\partial \vec{Z}}{\partial \vec{z}} \left(\vec{0} \right) \; - \; A_s$$

Lemma 3: *The quasi-linearization of the map (110) exists, is unique, is a linear canonical map and is defined by the matrix*

$$A_s \; = \; \mathrm{diag} \left(\frac{\partial \vec{X}}{\partial \vec{x}} \left(\vec{0}, \, \vec{0} \right), \; \frac{\partial \vec{S}}{\partial \vec{s}} \left(\vec{0}, \, \vec{0} \right) \right) \tag{111}$$

Remark: The equality (111) can be used as definition of quasi-linearization. In this case definition 2 becomes a statement which must be proven.

6.1.2 Factorization Theorem

The use of the change of coordinates in the form of the power series (110) is not too convenient for our purpose because the proof of canonicity is reduced to the checking of an infinite number of conditions for the coefficients of the Taylor expansion of the vector function $\vec{Z}(\vec{z})$. Dragt and Finn [18] have shown that any symplectic map in the neighborhood of a fixed point may be represented in the form of a composition of a linear symplectic map and a sequence of displacements along trajectories of the Hamilton system with Hamiltonians which are homogeneous polynomials of powers $3, 4, 5, \ldots$. A similar factorization is applicable in our case too, if instead of homogeneous polynomials we use Hamiltonians from the classes $\mathcal{H}_s(m)$.

Definition 3: *We will say that the function* $V(\vec{z}) \; \in \; \mathcal{O}_s(m)$ *, if*

$$\exists \; \lim_{t \to 0} \; \frac{1}{t^m} \, V(t \, \vec{x}, \, t^2 \, \vec{s}) \; \in \; \mathcal{H}_s(m)$$

Factorization Theorem A: *For every canonical map of the form (110) and for every integer* $m \geq 3$ *functions* $F_k(\vec{z}) \in \mathcal{H}_s(k) \;\; (3 \leq k \leq m)$ *can be found, such that*

$$\vec{Z}(\vec{z}) \; =_m : A_s : \exp(: F_3 :) \ldots \exp(: F_m :) \, \vec{z}$$

[9] This means that for an arbitrary $m = 0, 1, 2, \ldots$ and for an arbitrary $P(\vec{z}) \in \mathcal{H}_s(m)$ we have that $P(A_s \, \vec{z}) \; \in \; \mathcal{H}_s(m)$.

Here A_s is the matrix of the quasi-linearization of the map (110) and the symbol '$=_m$' means that the difference of the right and left parts is a function from the class $\mathcal{O}_s(m)$.

Here the exponential Lie operators are defined as usual as a power series (see, for example [19]) involving the spin-orbit Poisson bracket (18), and the action of the operator $: A_s :$ is defined by means of the rules

$$: A_s : \ f(\vec{z}) \ = \ f(: A_s : \vec{z})$$

for a smooth function f, and

$$: A_s : \ \vec{z} \ = \ A_s \, \vec{z}$$

for the identity mapping $\vec{z}$.

The proof of the factorization theorem A is outlined in Appendix C.

Applying theorem A to the canonical map $\vec{Z}^{-1}(\vec{z})$ and then inverting the factorization obtained we get

Factorization Theorem B: *For every canonical map of the form (110) and for every integer $m \geq 3$ functions $G_k(\vec{z}) \in \mathcal{H}_s(k)$ $(3 \leq k \leq m)$ can be found, such that*

$$\vec{Z}(\vec{z}) \ =_m \ \exp(: G_m :) \ldots \exp(: G_3 :) : A_s : \ \vec{z}$$

Remark: The classical Poisson bracket of two homogeneous polynomials of degree m and l is a homogeneous polynomial of degree $m + l - 2$. The same property holds for functions from classes $\mathcal{H}_s(m)$ and $\mathcal{H}_s(l)$ with respect to the spin-orbit Poisson bracket.

6.2 General Description of the Normal Form Algorithm

Consider a periodic real Hamiltonian function $H(\tau, \vec{z})$ of period 2π with respect to τ which in a neighbourhood of the point $\vec{z} = \vec{0}$ can be expanded in quasi-homogeneous polynomials in $\vec{z}$ beginning with degree 2

$$H(\tau, \vec{z}) \ = \ H_2(\tau, \vec{z}) \ + \ H_3(\tau, \vec{z}) \ + \ldots + \ H_k(\tau, \vec{z}) \ + \ldots \qquad (112)$$

The purpose of this subsection is to find the simplest possible form (**normal form**) to which the Hamiltonian function (112) can be reduced by means of a nonlinear canonical coordinate transformation of the type (110) which depends periodically on τ as a parameter.

6.2.1 Extension of Phase Space

Introduce two additional variables E, v and define the Poisson bracket and the new Hamiltonian by the rules:

$$\mathcal{H}(v,\, E,\, \vec{z}\,) \;=\; H(v,\, \vec{z}\,) \;-\; E$$

$$\{E,\, v\} \;=\; 1, \quad \{v,\, z_i\} \;=\; \{E,\, z_i\} \;=\; 0, \quad i \;=\; 1, 2, \ldots, 2n+3$$

According to the factorization theorem B, for any given truncation order m every canonical transformation of coordinates of the type (110), depending periodically on τ as a parameter, can be represented with the required precision in the form of an operator

$$\prod_{k=m}^{3} \exp(:\, G_k(\tau,\, \vec{z}\,) \,:) : A_s(\tau) : \tag{113}$$

In order to extend (113) to the canonical transformation in the extended phase space, we rewrite (113) in the form:

$$\prod_{k=m}^{3} \exp(:\, G_k(v,\, \vec{z}\,) \,:) : A_s(v) : \tag{114}$$

and define the action of the operator $: A_s(v) :$ on the identity mapping $(v,\, \vec{z},\, E)$ by the rules

$$: A_s(v) : v \;=\; v, \quad : A_s(v) : \vec{z} \;=\; A_s(v)\, \vec{z}$$

$$: A_s(v) : E \;=\; E \;-\; \frac{1}{2}\, A_{orbt}^{\top}\, J\, A_{orbt}^{'}\, \vec{x} \cdot \vec{x} \;+\; \frac{1}{2}\, \vec{s} \cdot \mathrm{curl}_{\vec{s}}\, (A_{spin}^{\top}\, A_{spin}^{'}\, \vec{s}\,)$$

Here

$$A_s(v) \;=\; \mathrm{diag}\, (A_{orbt}(v),\, A_{spin}(v))$$

$$A_{orbt}^{'} \;=\; \frac{dA_{orbt}}{dv}, \qquad A_{spin}^{'} \;=\; \frac{dA_{spin}}{dv}$$

and the $2n \times 2n$ matrix J is the symplectic unit matrix.

The exponential Lie operators are defined again by their power series with the help of the extended Poisson bracket for which we will keep the same notation $\{*, *\}$.

Now, informally speaking, the problem of finding the normal form can be expressed as follows: Find the simplest quasi-homogeneous polynomials $\bar{H}_k \in \mathcal{H}_s(k)$ $(2 \leq k \leq m)$ such that the initial Hamiltonian function $\mathcal{H}$ can be reduced to the Hamiltonian

$$\bar{\mathcal{H}} \;=\; \sum_{k=2}^{m} \bar{H}_k(v,\, \vec{z}) \;+\; \tilde{H}_{>m}(v,\, \vec{z}) \;-\; E, \quad \tilde{H}_{>m} \in \mathcal{O}_s(m+1)$$

by means of a nonlinear canonical coordinate transformation, which is defined by the operator (114). Here informal means that we have not yet defined the exact sense of the word ″simplest″. The usual way to do this is to introduce the simplest form for the quasi-quadratic polynomial $\bar{H}_2$ axiomatically and then to define $\bar{H}_k$ as functions which satisfy the condition $\{\bar{H}_k,\, \bar{H}_2 - E\} = 0$.

6.2.2 Linear Normalization

First we wish to simplify the quasi-quadratic part H_2 of the Hamiltonian $\mathcal{H}$ by means of the canonical transformation $: A_s :$. Denote

$$H_2 \;=\; \frac{1}{2}\, P(v)\, \vec{x} \cdot \vec{x} \;+\; \vec{w}(v) \cdot \vec{s}$$

where $P(v)$ is a real symmetric matrix of order $2n$ and $P(v + 2\pi) \equiv P(v)$, and where the real vector $\vec{w}(v)$ satisfies $\vec{w}(v + 2\pi) \equiv \vec{w}(v)$.

Applying the operator $: A_s(v) :$ to the Hamiltonian $\mathcal{H}$ we get the following formula for the quasi-quadratic part $\bar{H}_2$ of the new Hamiltonian $\mathcal{H}_{new} = : A_s : \mathcal{H}$

$$\bar{H}_2 \;=\; \frac{1}{2}\, A_{orbt}^{\top}\, (P A_{orbt} \;+\; J A'_{orbt})\, \vec{x} \cdot \vec{x} \;+$$

$$+\; \frac{1}{2}\, \vec{s} \cdot \operatorname{curl}_{\vec{s}} \left(A_{spin}^{\top} \left(C(\vec{w})\, A_{spin} \;-\; A'_{spin} \right) \vec{s} \right)$$

Below we will study the most important case for applications (to which we will refer later on as the **orbital elliptical case**) when all the eigenvalues of the one-turn revolution matrix of the system

$$\frac{d\vec{x}}{dv} \;=\; J P(v)\, \vec{x}$$

are distinct and lie on the unit circle (the general algorithm will be published elsewhere). Combining the results of the previous section and the well known normal form theory of classical linear Hamiltonian systems (see, for example [20, 21]) we are able to find real matrices $A_{spin} \in \mathrm{SO}(3)$ and $A_{orbt} \in \mathrm{Sp}(2n)$ 2π-periodic in v such that $\bar{H}_2$ finally takes the form

$$\bar{H}_2 \;=\; \frac{æ_k}{2}\left(q_k^2 \,+\, p_k^2\right) \,+\, \lambda_s \cdot s_3$$

where the values $æ_k$ and λ_s are called the **characteristic frequencies** (or the linear orbital and spin tunes).

6.2.3 Recursive Loop of Nonlinear Normalization

Let the functions $\bar{H}_k$ and G_k be already defined for $k = 3, \ldots, l-1$. Applying the operator $\exp(:G_l:)$ to the Hamiltonian

$$\bar{H}_2 \,+\, \ldots \,+\, \bar{H}_{l-1} \,+\, \tilde{H}_l \,+\, \ldots \,-\, E \;\stackrel{\mathrm{def}}{=}$$

$$\exp(:G_{l-1}:) \ldots \exp(:G_3:) : A_s : \mathcal{H}(v, E, \vec{z}\,)$$

and collecting the remainders from the class $\mathcal{H}_s(l)$, we get the equation for obtaining the functions $\bar{H}_l$ and G_l

$$\frac{\partial G_l}{\partial v} \,+\, \{G_l,\, \bar{H}_2\} \;=\; \bar{H}_l \,-\, \tilde{H}_l \tag{115}$$

This is the so-called **homology equation**. We say that the quasi-homogeneous polynomial G_l takes the quasi-homogeneous polynomial $\tilde{H}_l$ into the quasi-homogeneous polynomial $\bar{H}_l$ if (115) holds.

In order to solve the homology equation (115) we introduce complex coordinates $\vec{w} = (\vec{\eta},\, \vec{\xi},\, \vec{u}\,)$ related to the old coordinates $\vec{z} = (\vec{q},\, \vec{p},\, \vec{s}\,)$ by means of a **standard linear transformation** $\vec{z} = Q\vec{w}$:

$$q_k \;=\; \frac{1+i}{2}\,(\eta_k \,+\, \xi_k), \qquad p_k \;=\; -\frac{1-i}{2}\,(\eta_k \,-\, \xi_k)$$

$$k \;=\; 1, \ldots, n$$

$$s_1 \;=\; \frac{1+i}{2}\,(u_1 \,+\, u_2), \qquad s_2 \;=\; -\frac{1-i}{2}\,(u_1 \,-\, u_2)$$

$$s_3 \;=\; u_3$$

It is easy to verify that the matrix Q has the following properties:

$$QQ^* \;=\; I, \qquad Q^2 \;=\; -Q^*$$

where the asterisk $''*''$ indicates complex conjugation of a matrix.

This is a Poisson transformation, but it is not canonical (although it is symplectic with respect to orbital variables). The nonzero Poisson brackets of the new basis functions are given by the equalities:

$$\{\eta_k,\, \xi_k\} \;=\; 1, \quad \{u_1,\, u_2\} \;=\; u_3, \quad \{u_1,\, u_3\} \;=\; -iu_1, \quad \{u_2,\, u_3\} \;=\; iu_2$$

In the new coordinates the equation (115) has the form

$$\frac{\partial g_l}{\partial v} \;+\; \{g_l,\, \bar{h}_2\} \;=\; \bar{h}_l \;-\; \tilde{h}_l \tag{116}$$

where we have used the notation

$$\bar{h}_2(\vec{w}\,) \;=\; \bar{H}_2(Q\vec{w}\,) \;=\; i\text{æ}_k\eta_k\xi_k \;+\; \lambda_s u_3, \quad \tilde{h}_l(v,\, \vec{w}\,) \;=\; \tilde{H}_l(v,\, Q\vec{w}\,)$$

$$g_l(v,\, \vec{w}\,) \;=\; G_l(v,\, Q\vec{w}\,), \qquad \bar{h}_l(v,\, \vec{w}\,) \;=\; \bar{H}_l(v,\, Q\vec{w}\,)$$

Expanding the coefficients of the monomials in the polynomials g_l, $\tilde{h}_l$ and $\bar{h}_l$ in Fourier series with respect to v [10]

$$\begin{cases} g_l \;=\; g_l^{NIJL} \exp(iNv)\, \eta^I \xi^J u^L \\ \bar{h}_l \;=\; \bar{h}_l^{NIJL} \exp(iNv)\, \eta^I \xi^J u^L \\ \tilde{h}_l \;=\; \tilde{h}_l^{NIJL} \exp(iNv)\, \eta^I \xi^J u^L \end{cases} \tag{117}$$

(here the summation is made over all integers N and over all nonnegative integer vectors $I, J \in R^n$, $L \in R^3$ satisfying the condition $|I|+|J|+2|L| = l$[11]) and substituting them in the equation (116) we obtain a system of equations for the coefficients:

$$i \cdot [N + (I - J) \cdot \vec{\text{æ}} + (L_2 - L_1) \cdot \lambda_s] \cdot g_l^{NIJL} \;=\; \bar{h}_l^{NIJL} - \tilde{h}_l^{NIJL} \tag{118}$$

[10] We assume that this Fourier series converges absolutely.

[11] Here the function $|*|$ for the integer vector $\vec{k} \in R^m$ is defined as $|\vec{k}| = |k_1| + \ldots + |k_m|$.

Definition 4: *The characteristic frequencies* $æ_1, \ldots, æ_n, \lambda_s$ *satisfy a* **resonance relation of order** K *if there exist integers* k_l *not all equal to zero such that*

$$k_1 æ_1 + \ldots + k_n æ_n + k_{n+1}\lambda_s = 0 \ (mod\ 1) \tag{119}$$

$$\mid k_1 \mid + \ldots + \mid k_n \mid + 2\cdot \mid k_{n+1} \mid = K \tag{120}$$

The number of the linearly independent integer solutions $k_1, \ldots, k_{n+1}$ of equation (119) is an important characteristic of the frequencies $æ_1, \ldots, æ_n, \lambda_s$, and we will call it the **multiplicity of the resonance**.

Note that our definition of the order of a resonance (multiplier 2 in front of k_{n+1} in (120)) is different from the usual one, and corresponds to the definition of quasi-homogeneity.

Definition 5: *A quasi-homogeneous polynomial*

$$\bar{h}_l(v, \vec{w}) = \bar{h}_l^{NIJL} \exp(iNv)\, \eta^I \xi^J u^L$$

is said to be a **complex normal form** *if*

$$\mid N + (I - J)\cdot \vec{æ} + (L_2 - L_1)\cdot \lambda_s \mid \cdot \mid \bar{h}_l^{NIJL} \mid = 0$$

that is, the normal form contains only **resonant terms**.

In each quasi-homogeneous polynomial

$$h_l = h_l^{NIJL} \exp(iNv)\, \eta^I \xi^J u^L \tag{121}$$

we isolate the **resonant part** h_l', which contains all those and only those terms of (121) whose indices satisfy

$$N + (I - J)\cdot \vec{æ} + (L_2 - L_1)\cdot \lambda_s = 0$$

A quasi-homogeneous polynomial $h_l'' = h_l - h_l'$ is called the **nonresonant part** of the quasi-homogeneous polynomial h_l. Obviously, $h_l = h_l' + h_l''$. We also introduce the resonant H_l' and nonresonant H_l'' parts of a quasi-homogeneous polynomial $H_l(v, \vec{z})$ as

$$H_l'(v, \vec{z}) = h_l'(v, Q^*\vec{z}), \qquad H_l''(v, \vec{z}) = h_l''(v, Q^*\vec{z})$$

where $h'_l(v,\,\vec{w}\,)$ and $h''_l(v,\,\vec{w}\,)$ are the resonant and nonresonant parts of the quasi-homogeneous polynomial $h_l(v,\,\vec{w}\,) \;=\; H_l(v,\,Q\vec{w}\,)$.

According to the definition 5, if the quasi-homogeneous polynomial $\bar{h}_l$ is a complex normal form, then $\bar{h}'_l \;=\; \bar{h}_l\,.$

Definition 6: *A quasi-homogeneous polynomial*

$$\bar{h}_l(v,\,\vec{w}\,) \;=\; \bar{h}_l^{NIJL}\exp(iNv)\,\eta^I\xi^J u^L$$

is said to be a **nonresonant complex normal form** *if*

$$||\;N\;|+|\;I-J\;|+|\;L_2-L_1\;||\cdot|\;\bar{h}_l^{NIJL}\;|=\;0$$

that is, the normal form contains only **trivial resonant terms**.

Definition 7: *A quasi-homogeneous polynomial $\bar{H}_l(v,\vec{z})$ is said to be a* **(nonresonant) normal form** *if after the standard transformation $\vec{z}=Q\vec{w}$ we obtain the quasi-homogeneous polynomial $\bar{h}_l(v,\vec{w})=\bar{H}_l(v,Q\vec{w})$ which is a (nonresonant) complex normal form.*

It is easy to see (comparing definitions 5 and 7, and equation (118)) that for every given quasi-homogeneous polynomial $\tilde{H}_l$ there is the quasi-homogeneous polynomial G_l which takes $\tilde{H}_l$ into the normal form $\bar{H}_l$. Here the nonresonant part G''_l and the normal form $\bar{H}_l$ are uniquely determined, and the resonant part G'_l can be specified arbitrarily.

Among the quasi-homogeneous polynomials G_l which take $\tilde{H}_l$ into the normal form $\bar{H}_l$ we select the one for which $G'_l \;=\; 0$ and we call it **basic**.

Remembering that the initial Hamiltonian function $\mathcal{H}$ is real (the values of the function are real for real values of arguments) and that after a real canonical transformation : $A_s(v)$: we again have the real Hamiltonian function : $A_s(v)$: $\mathcal{H}$, we wish to find conditions on the functions G_l that will guarantee the reality of the transformation from : $A_s(v)$: $\mathcal{H}$ to normal form $\bar{\mathcal{H}}$. Denote by $\hat{h}(v,\vec{w})$ the quasi-homogeneous polynomial in which all the coefficients of the powers of $\vec{w}$ are complex conjugates (as functions of v) of the corresponding coefficients in the quasi-homogeneous polynomial $h(v,\vec{w})$.

Lemma 4: *Suppose that in the homology equation (115) $\bar{H}_2$ and $\tilde{H}_l$ are real functions. Then the normal form $\bar{H}_l$ will be a real function and the necessary and sufficient conditions that the quasi-homogeneous polynomial G_l (which takes $\tilde{H}_l$ into the normal form $\bar{H}_l$) will be a real function are that its resonant part $G'_l(v,\,\vec{z}\,)$ satisfies the condition*

$$g'_l(v,\,\vec{w}\,) \;=\; \hat{g}'_l(v,\,-Q^*\vec{w}\,) \quad \textit{where} \quad g'_l(v,\,\vec{w}\,) \;=\; G'_l(v,\,Q\vec{w}\,) \qquad (122)$$

In particular, the basic solution of the homology equation G_l ($G'_l = 0$) which takes $\tilde{H}_l$ into the normal form $\bar{H}_l$ will be a real function.

6.2.4 Normalization Theorem

Now we are ready to summarize the results of this section as follows

Normalization Theorem: *In the orbital elliptical case, for a real initial Hamiltonian of the form*

$$\mathcal{H}(v,\, E,\, \vec{z}) \;=\; \sum_{k=2}^{\infty} H_k(v,\, \vec{z}) \;-\; E, \qquad H_k \;\in\; \mathcal{H}_s(k)$$

and for every integer $m \geq 3$ there exists a real canonical transformation depending periodically on v and defined by an operator

$$\prod_{k=m}^{3} \exp(:\, G_k(v,\, \vec{z}) \,:) \;:\, A_s(v) \,:$$

under which the initial Hamiltonian becomes a real Hamiltonian

$$\bar{\mathcal{H}}(v,\, E,\, \vec{z}) \;=\; \frac{æ_k}{2}\left(q_k^2 + p_k^2\right) \;+\; \lambda_s \cdot s_3 \;+\; \sum_{k=3}^{m} \bar{H}_k(v,\, \vec{z}) \;+\; \tilde{H}_{>m}(v,\, \vec{z}) \;-\; E$$

such that $\tilde{H}_{>m} \;\in\; \mathcal{O}_s(m+1)$ and all quasi-homogeneous polynomials $\bar{H}_k \;\in\; \mathcal{H}_s(k)$ are normal forms. Here the resonant parts of the quasi-homogeneous polynomials G_k may be chosen arbitrarily satisfying condition (122); then the nonresonant parts of G_k and $\bar{H}_k$ are uniquely defined.

7 General Properties of Systems whose Hamiltonians are Normal Forms

A Hamiltonian system whose Hamiltonian function is a normal form has, as a rule, an abundance of symmetries, and this enables the order to be lowered. This is the advantage of a normal form over an arbitrary Hamiltonian function. We will not discuss the technical details of the procedure of lowering the order here, but concentrate on the question of finding invariant functions (conservations laws). For convenience we will use the complex coordinates $\vec{w}$, but it is helpful to remember that in the original coordinates $\vec{z}$ all functions that we use will be functions with real values.

7.1 Normal Forms and Invariant Functions

Writing a normal form $\bar{\mathcal{H}}$ in complex coordinates $\vec{w}$ we have

$$\bar{h} = \bar{h}(v, E, \vec{w}) = \bar{\mathcal{H}}(v, E, Q\vec{w}) = i\text{æ}_k \eta_k \xi_k + \lambda_s u_3 + \sum_{k=3}^{\infty} \bar{h}_k - E \qquad (123)$$

where

$$\bar{h}_k \;=\; \bar{h}_k^{NIJL} \exp(iNv)\, \eta^I \xi^J u^L$$

and the summation is made over all integers N and over all nonnegative integer vectors I, J, L satisfying the conditions

$$|I| + |J| + 2|L| \;=\; k$$

$$N \;+\; (I - J) \cdot \vec{\text{æ}} \;+\; (L_2 - L_1) \cdot \lambda_s \;=\; 0$$

We denote by $\mathcal{L}$ the linear subspace of R^{n+1} that is the linear hull of all integer solutions of equation (119). The dimension of $\mathcal{L}$ is equal to the multiplicity of the resonance. Let a vector $\vec{m} \in R^{n+1}$ be orthogonal to $\mathcal{L}$, i.e. $\vec{m} \perp \mathcal{L}$. Then the function

$$F_{\vec{m}} \;=\; i\, m_k\, \eta_k\, \xi_k \;+\; m_{n+1}\, u_3 \qquad (124)$$

satisfies the condition $\{F_{\vec{m}}, \bar{h}\} \;=\; 0$ and is hence an invariant. So we have $n + 1 - \dim \mathcal{L}$ functionally independent invariants of the type (124).

Besides that an arbitrary normal form commutes with the function

$$F \;=\; \bar{h} \;-\; (i\, \text{æ}_k\, \eta_k\, \xi_k \;+\; \lambda_s\, u_3 \;-\; E) \qquad (125)$$

and with the Casimir function

$$s_1^2 \;+\; s_2^2 \;+\; s_3^2 \;=\; 2\, i\, u_1\, u_2 \;+\; u_3^2 \qquad (126)$$

Remark: And, of course, the Hamiltonian (123) is always a constant of motion (in extended phase space).

7.2 Action-Angle Type Coordinates

Instead of the standard cartesian coordinates

$$\eta_1, \ldots, \eta_n, \quad \xi_1, \ldots, \xi_n, \quad u_1, u_2, u_3$$

for normal form analysis and construction we can use action-angle type canonical polar coordinates

$$\rho_1, \ldots, \rho_{n+2}, \quad \varphi_1, \ldots, \varphi_{n+1} \tag{127}$$

introduced with the help of the formulae

$$\eta_l \;=\; -\frac{1+i}{2}\sqrt{2\rho_l}\exp(i\varphi_l), \qquad \xi_l \;=\; \frac{1+i}{2}\sqrt{2\rho_l}\exp(-i\varphi_l)$$

$$l \;=\; 1, \ldots, n$$

$$u_1 \;=\; \frac{1-i}{2}\sqrt{2\rho_{n+1}}\exp(-i\varphi_{n+1}), \qquad u_2 \;=\; \frac{1-i}{2}\sqrt{2\rho_{n+1}}\exp(i\varphi_{n+1})$$

$$u_3 \;=\; \rho_{n+2}$$

Note that this change of variables is not defined on the set

$$\eta_1 \cdot \ldots \cdot \eta_n \cdot \xi_1 \cdot \ldots \cdot \xi_n \cdot u_1 \cdot u_2 \;=\; 0$$

and, as in the case of the standard linear transformation Q, it is a Poisson transformation, but it is not canonical (and is not even symplectic with respect to the orbital variables[12]).

The nonzero Poisson brackets of the new basis functions are given by the equalities

$$\{\varphi_l,\, \rho_l\} \;=\; 1, \qquad l \;=\; 1, \ldots, n$$

$$\{\rho_{n+1},\, \varphi_{n+1}\} \;=\; \rho_{n+2}, \qquad \{\varphi_{n+1},\, \rho_{n+2}\} \;=\; 1$$

[12]The symplecticity with respect to the orbital variables can be easily achieved, if necessary, by interchanging the places of the orbital actions and angles in (127).

and the Hamilton equations with the Hamiltonian function $H(\vec{\rho},\,\vec{\varphi})$ take the form

$$\frac{d\rho_l}{dt} \;=\; -\frac{\partial H}{\partial \varphi_l}, \qquad \frac{d\varphi_l}{dt} \;=\; \frac{\partial H}{\partial \rho_l}$$

$$l \;=\; 1,\ldots,n$$

$$\frac{d\rho_{n+1}}{dt} \;=\; \rho_{n+2}\cdot\frac{\partial H}{\partial \varphi_{n+1}}, \qquad \frac{d\varphi_{n+1}}{dt} \;=\; -\rho_{n+2}\cdot\frac{\partial H}{\partial \rho_{n+1}} \;+\; \frac{\partial H}{\partial \rho_{n+2}}$$

$$\frac{d\rho_{n+2}}{dt} \;=\; -\frac{\partial H}{\partial \varphi_{n+1}}$$

The direct transition from the original variables $\vec{z}$ to the polar coordinates introduced above has the form

$$q_l \;=\; \sqrt{2\rho_l}\,\sin(\varphi_l), \qquad p_l \;=\; \sqrt{2\rho_l}\,\cos(\varphi_l)$$

$$l \;=\; 1,\ldots,n$$

$$s_1 \;=\; \sqrt{2\rho_{n+1}}\,\cos(\varphi_{n+1}), \qquad s_2 \;=\; \sqrt{2\rho_{n+1}}\,\sin(\varphi_{n+1})$$

$$s_3 \;=\; \rho_{n+2}$$

7.3 Normal Form in Nonresonant Case

In the **nonresonant case** ($\dim \mathcal{L} = 0$) the normal form (123) contains only those monomials for which

$$I \;=\; J, \quad L_1 \;=\; L_2, \quad N \;=\; 0$$

and hence we can rewrite it as

$$\bar{h} \;=\; i\,æ_k\,\eta_k\,\xi_k \;+\; \lambda_s\,u_3 \;+\; \bar{h}_{IL_1L_3}\,(\eta\xi)^I\,(u_1u_2)^{L_1}\,u_3^{L_3} \;-\; E \qquad (128)$$

where the summation is made over all nonnegative integer vectors I and over all nonnegative integers L_1, L_3 satisfying the condition

$$2|I| \,+\, 4L_1 \,+\, 2L_3 \,\geq\, 3$$

According to (124) the functions

$$i\eta_1\xi_1, \quad \ldots, \quad i\eta_n\xi_n, \quad u_3 \tag{129}$$

are constants of motion. Taking into account that $\bar{h}$ is independent of v, and subtracting from the Casimir function (126) the invariant u_3^2 we get two additional invariants

$$E, \quad iu_1u_2 \tag{130}$$

Altogether (129) and (130) give us $n+3$ independent integrals of motion. This allows us to consider the system with the Hamiltonian (128) as integrable. Moreover it is easy to find the general solution by quadrature. Let us do this in the original real variables.

In the variables $\vec{z}$ the Hamiltonian (128) can be written as

$$\bar{\mathcal{H}} \;=\; \bar{H}\,(J_1,\,\ldots,\,J_n,\,I_1,\,I_2) \;-\; E \tag{131}$$

where

$$J_k \;=\; \frac{1}{2}\left(q_k^2 + p_k^2\right), \quad I_1 \;=\; \frac{1}{2}\left(s_1^2 + s_2^2\right), \quad I_2 \;=\; s_3$$

are the functions (129), (130) expressed in variables $\vec{z}$, and the equations of motion take the form

$$\frac{dq_k}{d\tau} \;=\; \omega_k\, p_k, \qquad \frac{dp_k}{d\tau} \;=\; -\omega_k\, q_k \tag{132}$$

$$\frac{ds_1}{d\tau} \;=\; -\Omega\, s_2, \quad \frac{ds_2}{d\tau} \;=\; \Omega\, s_1, \quad \frac{ds_3}{d\tau} \;=\; 0 \tag{133}$$

Here we have used the notations

$$\omega_k\,(J_1,\,\ldots,\,J_n,\,I_1,\,I_2) \;=\; \frac{\partial \bar{H}}{\partial J_k}, \tag{134}$$

$$\Omega\,(J_1,\,\ldots,\,J_n,\,I_1,\,I_2)\;=\;\frac{\partial\bar{H}}{\partial I_2}\;-\;\frac{\partial\bar{H}}{\partial I_1}\cdot I_2 \tag{135}$$

and we have omitted the trivial equations for the variables v and E.

Fixing for simplicity $\tau_0 = 0$ and taking into account that the functions (134) and (135) are constants of motion, we get the solution of the system (132), (133) in the form

$$q_k(\tau)\;=\;\cos(\omega_k^0\tau)\,q_k(0)\;+\;\sin(\omega_k^0\tau)\,p_k(0)$$

$$p_k(\tau)\;=\;-\sin(\omega_k^0\tau)\,q_k(0)\;+\;\cos(\omega_k^0\tau)\,p_k(0)$$

$$s_1(\tau)\;=\;\cos(\Omega^0\tau)\,s_1(0)\;-\;\sin(\Omega^0\tau)\,s_2(0)$$

$$s_2(\tau)\;=\;\sin(\Omega^0\tau)\,s_1(0)\;+\;\cos(\Omega^0\tau)\,s_2(0)$$

$$s_3(\tau)\;=\;s_3(0)$$

where ω_k^0 and Ω^0 are the values of the functions (134) and (135) calculated at the initial time $\tau = 0$.

Obviously, these formulae give us the solution of the triangular system too, if we redefine ω_k and Ω in (134), (135) as

$$\omega_k\,(J_1,\ldots,J_n)\;=\;\left.\frac{\partial\bar{H}}{\partial J_k}\right|_{I_1=I_2=0},\quad \Omega\,(J_1,\ldots,J_n)\;=\;\left.\frac{\partial\bar{H}}{\partial I_2}\right|_{I_1=I_2=0}$$

The values ω_k and Ω are called the (**nonlinear**) **orbital** and **spin tunes** respectively.

7.4 Normal Form in the Case of a Single Isolated Resonance

We shall say that the Hamiltonian function $\bar{\mathcal{H}}$ is a **generalized single resonance normal form**[13] if there exist integers

$$k_1^0,\quad \ldots,\quad k_{n+1}^0,\quad N^0$$

[13]In order to keep the size of this paper within reasonable bounds we shall not discuss the formal aspects of constructing of the real canonical transformation which will bring the initial Hamiltonian to a single resonance normal form because this procedure is almost the same as that described in subsection 6.2.

where not all k_l^0 are zero such that when written in complex coordinates $\vec{w}$ this function has the form

$$\bar{h} \;=\; i\,æ_k\,\eta_k\,\xi_k \;+\; \lambda_s\,u_3 \;+\; \bar{h}^{NIJL}\exp(iNv)\,\eta^I\xi^J u^L \;-\; E \tag{136}$$

and so that in the expansion (136) there are only those terms whose indices for some integer $m \;=\; m(N,I,J,L)$ satisfy

$$I_l \;-\; J_l \;=\; m\cdot k_l^0, \qquad l \;=\; 1,\ldots,n$$

$$L_2 \;-\; L_1 \;=\; m\cdot k_{n+1}^0, \quad N \;=\; m\cdot N^0, \quad |I|+|J|+2|L| \;\geq\; 3$$

Thus, the normal form (136) contains only those monomials for which

$$N \;+\; (I-J)\cdot \vec{æ} \;+\; (L_2-L_1)\cdot\lambda_s \;=$$

$$=\; m\cdot\Big(N^0 \;+\; k_1^0\cdot æ_1 \;+\; \ldots \;+\; k_n^0\cdot æ_n \;+\; k_{n+1}^0\cdot\lambda_s\Big)$$

and we do not require the condition

$$N^0 \;+\; k_1^0\cdot æ_1 \;+\; \ldots \;+\; k_n^0\cdot æ_n \;+\; k_{n+1}^0\cdot\lambda_s \;=\; 0$$

to be satisfied.

Let a vector $\vec{m}\in R^{n+1}$ be orthogonal to the vector $\vec{k}^0$. Then the function

$$F_{\vec{m}} \;=\; i\,m_k\,\eta_k\,\xi_k \;+\; m_{n+1}\,u_3 \tag{137}$$

is an invariant. So, together with the Casimir function (126) and the Hamiltonian (136), we have $n+2$ functionally independent integrals of motion, and in the remainder of this subsection we shall find an additional invariant that will allow us to consider the single resonance problem as integrable.

7.4.1 Single Resonance Between Orbital Tunes

If we have a single resonance between orbital tunes (i.e. $k_{n+1}^0 \;=\; 0$) then the required additional invariant is given by an expression

$$F \;=\; i\,\varepsilon\,k_k^0\,\eta_k\,\xi_k \;+\; \Big(\bar{h} \;+\; E \;-\; i\,æ_k\,\eta_k\,\xi_k \;-\; \lambda_s\,u_3\Big) \tag{138}$$

where the quantity

$$\varepsilon \;=\; \frac{æ_k k_k^0 \;+\; N^0}{\parallel \vec{k}^0 \parallel^2}$$

is called the **distance from orbital resonance**. If the integral (138) is independent of the integrals of the form (137) and of the Casimir function (126), the single resonance normal form (136) is completely integrable. If not, then $\bar{h} + E$ is a series in $i\eta_1\xi_1, \ldots, i\eta_n\xi_n, iu_1u_2, u_3$ only, and the additional integral will be $F = ik_k^0\eta_k\xi_k$. Hence, we are always in the situation of complete integrability.

The triangular truncated equations of spin motion corresponding to the Hamiltonian (136)

$$\frac{du_1}{d\tau} = -i\Omega u_1, \quad \frac{du_2}{d\tau} = i\Omega u_2, \quad \frac{du_3}{d\tau} = 0, \quad \Omega = \left.\frac{\partial \bar{h}}{\partial u_3}\right|_{u_1=u_2=u_3=0} \tag{139}$$

can be easily integrated by quadrature if we know the solution of the orbital part of equations of motion. In the original variables the system (139) will take the form

$$\frac{ds_1}{d\tau} = -\Omega s_2, \quad \frac{ds_2}{d\tau} = \Omega s_1, \quad \frac{ds_3}{d\tau} = 0, \quad \Omega = \left.\frac{\partial \bar{\mathcal{H}}}{\partial s_3}\right|_{s_1=s_2=s_3=0}$$

and this allows us to draw a number of conclusions. For example, we can state that in principle, it is possible to organize slow extraction of a polarized proton beam using a third-integer orbital resonance without significant loss of polarization.

Remark: Note that the case considered here includes, in our opinion, the correct treatment of the situation when the synchrotron frequency is very small compared to the other frequencies in the system (as, for example, in the case of HERA-p ring [11]).

7.4.2 Single Spin-Orbit Resonance

If $k_{n+1}^0 \neq 0$ then as an additional invariant we can take the function

$$F \;=\; \varepsilon\, u_3 \;+\; \left(\bar{h} \;+\; E \;-\; i\, æ_k\, \eta_k\, \xi_k \;-\; \lambda_s\, u_3\right) \tag{140}$$

where the quantity

$$\varepsilon \;=\; \frac{æ_k\, k_k^0 \;+\; \lambda_s\, k_{n+1}^0 \;+\; N^0}{k_{n+1}^0}$$

is called the **distance from spin-orbit resonance**. If the integral (140) is independent of the integrals of the form (137) and of the Casimir function (126), the single resonance normal form (136) is completely integrable. If not, then $\bar{h} + E$ is a series in $i\eta_1\xi_1, \ldots, i\eta_n\xi_n, iu_1u_2, u_3$ only, and the additional integral will be $F = u_3$. Hence, again we are in the situation of complete integrability.

In more detail the invariant (140) can be written as

$$F(v,\, \vec{\eta},\, \vec{\xi},\, \vec{u}\,) \;=\; \varepsilon\, u_3 \;+\; \bar{h}^{NIJL} \exp(iNv)\, \eta^I \xi^J u^L$$

and we see that in general it does not satisfy the condition $F(v, \vec{\eta}, \vec{\xi}, \vec{0}) \equiv 0$, which is very important for application to the triangular system. This situation can be easily changed if we choose n invariants of the type (137) in the form

$$V_1 \;=\; i\eta_1\xi_1 - \frac{k_1^0}{k_{n+1}^0}\, u_3, \qquad \ldots, \qquad V_n \;=\; i\eta_n\xi_n - \frac{k_n^0}{k_{n+1}^0}\, u_3 \tag{141}$$

and instead of F use the invariant

$$\bar{F} \;=\; F \;-\; \sum_{|I|>1} \bar{h}^{0II0}\, i^{-|I|}\, V^I$$

So the triangular system will have the spin dependent integral of motion

$$\left.\frac{\partial \bar{F}}{\partial u_1}\right|_{\vec{u}=\vec{0}} \cdot u_1 \;+\; \left.\frac{\partial \bar{F}}{\partial u_2}\right|_{\vec{u}=\vec{0}} \cdot u_2 \;+\; \left.\frac{\partial \bar{F}}{\partial u_3}\right|_{\vec{u}=\vec{0}} \cdot u_3$$

and hence the solution for spin can be obtained by quadrature following, for example, the procedure described in Appendix D (a detailed analysis will be published in a separate paper).

Remark: The set of invariants (141) show us that a proton beam cannot be split into two polarized parts with different amplitudes using the classical Stern-Gerlach effect at a single spin-orbit resonance as has been suggested in [34]. For related comments see [33, 35].

8 Methods of Numerical Integration

Today **symplectic tracking** methods for orbital motion (methods which conserve the classical Poisson bracket[14]) are common tools in accelerator physics and we believe that such methods, which conserve the Poisson bracket (18) exactly or with high accuracy, are the most suitable ones for numerical simulation of the equations (21)-(23). For simplicity we consider the case of autonomous Hamiltonian functions. In the nonautonomous case we can repeat all the steps, introducing two additional canonical variables to obtain an autonomous Hamiltonian system in a higher dimensional phase space.

Those methods, of course, are mainly of theoretical interest, but in application to the triangular system (practical interest) the Hamiltonian approach allows us to reduce the initial problem of numerical integration of the system (1)-(2) to that of symplectic integration of the equations of orbital motion (1) only.

8.1 Canonical Numerical Methods for Hamiltonian Equations

Let K be a compact subset of the manifold M, and the autonomous Hamiltonian function $H(\vec{z})$ be zero onto the set $M \setminus N$ (here N is some compact subset of M containing K). Then the solutions of the Hamiltonian system (11) generate the one-parameter group of the canonical transformations of phase space

$$T(\tau) : M \ \rightarrow \ M, \qquad T(0) \ = \ I$$

Consider a one-parameter family of canonical transformations

$$T_m(\tau) : M \ \rightarrow \ M$$

defined for $\mid \tau \mid < \ \tau_0 \ \ (0 \ < \ \tau_0)$.

We will say that $T_m(\tau)$ **approximates** $T(\tau)$ in a neighbourhood of the identity mapping with the order m (with respect to K), if for all $\vec{z} \in K$

$$\| \ T(\tau) \, \vec{z} \ - \ T_m(\tau) \, \vec{z} \ \| = \ O(\tau^{m+1})$$

[14]Numerical integration schemes for differential equations which preserve other **qualitative properties** (like autonomous Hamiltonian or, more generally, first integral conserving algorithms, and so forth) are discussed in [22], and more recent references may be found in [23]

Here $\| \cdot \|$ is the norm in Euclidean space, containing the manifold M.

A one-parameter family $T_m(\tau)$ is said to be **locally computable** if for each $\vec{z}_0 \in K$ there is a chart U (from the atlas of the manifold M) containing all points

$$\vec{z}_\tau \;=\; T_m(\tau)\,\vec{z}_0$$

for $|\,\tau\,| < \tau_0$ and a vector function $\vec{F}_U$ such that $\vec{z}_\tau$ is defined uniquely by the equation

$$\vec{F}_U(\tau,\, \vec{z}_0,\, \vec{z}_\tau) \;=\; \vec{0}$$

Definition: *A locally computable one-parameter family of canonical transformations $T_m(\tau)$ is said to be a* **canonical integrator** *of order m of the group $T(\tau)$, if $T_m(\tau)$ approximates $T(\tau)$ with the order m.*

8.1.1 Recursively Generated High Order Canonical Integrators

Let us assume that by some method we find a canonical integrator of order m of the group T, which has in each of the charts U the formal representation[15]

$$T_m(\tau) \;=\; \exp(:\,-\tau\,H \;+\; \tau^{m+1}R_m(\tau,\,U)\,:) \;+\; O(\tau^{r+1}) \tag{142}$$

where $m < r$.

The purpose of the present section is to demonstrate a general approach for constructing higher order integrators starting from integrator (142) of order m.

Following [24, 25] we consider the mapping

$$P(\tau) \;=\; \prod_{j=1}^{n} T_m^{q_j}(p_j\,\tau) \tag{143}$$

where the exponents q_j are nonzero integers and the coefficients p_j are real numbers. Substituting the representation (142) in (143) and combining

[15] The representation (142) holds if, for example, we assume the constancy of the rank of the Poisson bracket, the possibility of introducing Darboux coordinates globally in each of the charts U and some smoothness properties of the integrator considered.

the product of exponential operators into a single Lie exponent using the Campbell-Baker-Hausdorff formula we have

$$P(\tau) \;=\; O(\tau^{r+1}) \,+$$

$$+\; \exp(:\, -\tau(q_j p_j)H \;+\; \tau^{m+1}(q_j p_j^{m+1})R_m(0,U) \;+\; O(\tau^{m+2})\,:) \qquad (144)$$

If $q_j\,,\; p_j$ satisfy the conditions

$$q_j \cdot p_j \;=\; 1, \qquad q_j \cdot p_j^{m+1} \;=\; 0 \qquad (145)$$

and $m+1 \;\geq\; r$ then from (144) it follows that the mapping (143) is a canonical integrator of order $m+1$ of the group $T(\tau)\,.$

Besides that, if m is an even number, $m+2 \;\geq\; r\,,\;\; T_m(\tau)$ is **time reversible**, so that

$$T_m(\tau)\, T_m(-\tau) \;=\; I$$

and

$$q_{n+1-j} \;=\; q_j, \qquad p_{n+1-j} \;=\; p_j$$

then (143) gives us a time reversible canonical integrator of order $m+2$.

Obviously, the scheme (143) may be recursively applied any number of times to obtain a canonical integrator of required order l (of course, with the assumption that this order l is not bigger than r).

Example : Let

$$a_m \;=\; \left(r_m \;+\; l_m \left(-\frac{r_m}{l_m}\right)^{\frac{1}{m+1}}\right)^{-1}, \qquad b_m \;=\; \left(-\frac{r_m}{l_m}\right)^{\frac{1}{m+1}} a_m$$

where nonzero integers $r_m,\, l_m\; (r_m \;\neq\; \pm l_m)$ for odd m satisfy the condition

$$r_m\, l_m \;<\; 0$$

Then

$$\begin{cases} r_m \cdot a_m \;\;\; + \;\; l_m \cdot b_m \;\;\; = \;\; 1 \\ \\ r_m \cdot a_m^{m+1} \;\; + \;\; l_m \cdot b_m^{m+1} \;\; = \;\; 0 \end{cases}$$

so that (145) is satisfied for $n = 2$ and hence the integrator of order $m+1$ can be chosen in the form

$$T_{m+1}(\tau) \;=\; T_m^{r_m}(a_m\,\tau)\,T_m^{l_m}(b_m\,\tau)$$

If additionally m and $|r_m|$ are even numbers, and $T_m(\tau)$ is time reversible, then the integrator of order $m+2$ can be taken as

$$T_{m+2}(\tau) \;=\; T_m^{\frac{r_m}{2}}(a_m\,\tau)\,T_m^{l_m}(b_m\,\tau)\,T_m^{\frac{r_m}{2}}(a_m\,\tau)$$

Remark : Note that the recursive scheme considered is slightly more general than that used in [24, 25]. Here it is not necessary that the integers q_j in (143) are positive and the initial integrator is not assumed to be time reversible.

8.1.2 Low Order Canonical Integrators for Equations of Classical Spin-Orbit Motion

In order to be able to start the recursive procedure described above, we now turn to the problem of constructing low order canonical integrators for a Hamiltonian system. We will give a few examples, and other helpful recipes (and references) may be found in [22, 26, 27, 28].

Example 1: If $\hat{J}$ the structure matrix of Poisson bracket is constant, then for any constant matrix A, satisfying the condition $\hat{J} = A\hat{J} + \hat{J}A^{\top}$, the mapping

$$\vec{z}_\tau \;=\; \vec{z}_0 \;+\; \tau\,\vec{\varphi}(A\,\vec{z}_\tau \;+\; (I-A)\,\vec{z}_0)$$

where

$$\vec{\varphi}(\vec{z}) \;=\; \hat{J}\;\mathrm{grad}_{\vec{z}}\,H(\vec{z})$$

is a canonical integrator of first order. In the special case when $A = \frac{1}{2}I$, this integrator is a second order approximation, is time reversible, and is known as the **mid-point rule** [22].

Example 2: The property that the integrator is canonical is preserved under Poisson transformations. This simple remark combined with example 1 allows us to construct first or second order canonical integrators for the equations of classical spin-orbit motion, because we know the explicit form of the transition formulae to Darboux coordinates. Note that although this

approach is applicable for arbitrary dependence of the Hamiltonian function on orbit and spin variables, its computer realization requires the use of more than one (at least two) local coordinate systems.

Example 3: Consider a Hamiltonian function which depends on spin variables only in the linear combination $\vec{i} \cdot \vec{s}$

$$H \;=\; H\left(\vec{x},\, \vec{i} \cdot \vec{s}\right)$$

Without loss of generality we can assume that $\left|\vec{i}\right| \;=\; 1$. Supplement the vector $\vec{i}$ with two unit vectors $\vec{j}$ and $\vec{k}$ satisfying the condition

$$\vec{i} \cdot \left[\, \vec{j} \times \vec{k} \,\right] \;=\; 1$$

so that the triplet $\vec{i}$, $\vec{j}$, $\vec{k}$ forms an orthogonal coordinate system.

Then the mapping given by the system of equations

$$\vec{s}_\tau \;=\; \begin{pmatrix} \vec{i} \\ \vec{j} \\ \vec{k} \end{pmatrix}^{\top} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\alpha\tau) & -\sin(\alpha\tau) \\ 0 & \sin(\alpha\tau) & \cos(\alpha\tau) \end{pmatrix} \begin{pmatrix} \vec{i} \\ \vec{j} \\ \vec{k} \end{pmatrix} \vec{s}_0$$

$$\vec{x}_\tau \;=\; \vec{x}_0 \;+\; \tau\, \vec{\varphi}\left(\frac{\vec{x}_\tau + \vec{x}_0}{2},\;\; \vec{i} \cdot \vec{s}_0\right)$$

is a time reversible canonical integrator of second order defined in the initial variables (global ones in 9-dimensional phase space). Here J is the symplectic unit and

$$\vec{\varphi}\left(\vec{x},\, \vec{i} \cdot \vec{s}\right) \;=\; J\, \mathrm{grad}_{\,\vec{x}}\, H\left(\vec{x},\, \vec{i} \cdot \vec{s}\right)$$

$$\omega\left(\vec{x},\, \vec{i} \cdot \vec{s}\right) \;=\; \vec{i} \cdot \mathrm{grad}_{\,\vec{s}}\, H\left(\vec{x},\, \vec{i} \cdot \vec{s}\right)$$

$$\alpha \;=\; \omega\left(\frac{\vec{x}_\tau + \vec{x}_0}{2},\; \vec{i} \cdot \vec{s}_0\right)$$

Example 4: If the Hamiltonian function $H(\vec{x},\, \vec{s})$ can be represented in the form of the sum

$$H(\vec{x},\, \vec{s}) \;=\; H_1\left(\vec{x},\, \vec{i}_1 \cdot \vec{s}\right) \;+\; \ldots \;+\; H_m\left(\vec{x},\, \vec{i}_m \cdot \vec{s}\right)$$

then the mapping

$$A_{H_m}^{\vec{i}_m}\left(\frac{\tau}{2}\right)\ldots A_{H_2}^{\vec{i}_2}\left(\frac{\tau}{2}\right)A_{H_1}^{\vec{i}_1}(\tau)\,A_{H_2}^{\vec{i}_2}\left(\frac{\tau}{2}\right)\ldots A_{H_m}^{\vec{i}_m}\left(\frac{\tau}{2}\right)$$

where $A_H^{\vec{i}}$ is the integrator of example 3, gives us a time reversible canonical integrator of second order. It solves (at least in theory) the problem of canonical integration for Hamiltonians linearly dependent on spin.

8.2 Approximately Canonical Numerical Methods for Short-Term Tracking (Numerical Methods for the Triangular System)

The length of the vector $\vec{s}$ is commensurate with Planck's constant $\hbar$. If we perform the renormalization

$$\vec{s}_{old} \;\rightarrow\; \frac{\hbar}{2}\cdot\vec{s}_{new}, \qquad |\,\vec{s}_{new}\,| = 1$$

then the equations of spin-orbit motion corresponding to the Hamiltonian function $H(\vec{x},\,\vec{s})$ become

$$\frac{d\vec{x}}{d\tau} \;=\; J\;\mathrm{grad}_{\vec{x}}\,H_{orbt}(\vec{x}) \;+\; O(\hbar)$$

$$\frac{d\vec{s}_{new}}{d\tau} \;=\; \left[\,\vec{W}(\vec{x}\,)\times\vec{s}_{new}\,\right] \;+\; O(\hbar)$$

where

$$H_{orbt}(\vec{x}) \;=\; H\left(\vec{x},\,\vec{0}\right) \quad \text{and} \quad \vec{W}(\vec{x}) \;=\; (\mathrm{grad}_{\vec{s}}\,H(\vec{x},\,\vec{s}\,))|_{\vec{s}=\vec{0}}$$

Consequently both the effect of spin on the orbit motion and the induced nonlinear influence of spin on itself, are very small and we will neglect them, if the integration interval T satisfies the condition

$$T \;<<\; \hbar^{-1}$$

This means that with high accuracy we can reduce the initial problem of numerical integration of the equations of spin-orbit motion to that of symplectic

integration of orbital motion equations only. We will now demonstrate this in a few steps.

First, for any vector $\vec{U}(\vec{x})$ we can approximate the action of the mapping

$$\exp\left(-:\vec{U}(\vec{x})\cdot\vec{s}\ +\ O\left(|\vec{s}|^2\right):\right)$$

on phase space variables with required precision by means of the explicit formulae

$$\vec{x}_f\ =\ \vec{x}_i \tag{146}$$

$$\vec{s}_f\ =\ \left(I\ +\ \frac{\sin\left(|\ \vec{U}\ |\right)}{|\ \vec{U}\ |}\,A(\vec{x}_i)\ +\ \frac{1\ -\ \cos\left(|\ \vec{U}\ |\right)}{|\ \vec{U}\ |^2}\,A^2(\vec{x}_i)\right)\ \vec{s}_i \tag{147}$$

where

$$A\ =\ \begin{pmatrix} 0 & -U_3 & U_2 \\ U_3 & 0 & -U_1 \\ -U_2 & U_1 & 0 \end{pmatrix}$$

Second, for any given positive integer k and using the Campbell-Baker-Hausdorff formula it is possible to find vectors

$$\vec{W}^1(\vec{x},\,\tau),\qquad \vec{W}^2(\vec{x},\,\tau),\qquad \vec{W}^3(\vec{x},\,\tau),\qquad \vec{W}^4(\vec{x},\,\tau)$$

so that the following decomposition formulae obtain

$$\exp\left(-:\tau H:\right)\ =\ \exp\left(-:\tau\left(H_{orbt}+\vec{W}\cdot\vec{s}+O\left(|\vec{s}|^2\right)\right):\right)\ =^k$$

$$\exp\left(-:\frac{\tau}{2}H_{orbt}:\right)\exp\left(-:\tau\left(\vec{W}^1\cdot\vec{s}+O\left(|\vec{s}|^2\right)\right):\right)\exp\left(-:\frac{\tau}{2}H_{orbt}:\right)=^k$$

$$=^k\ \exp\left(-:\frac{\tau}{2}\left(\vec{W}^2\cdot\vec{s}+O\left(|\vec{s}|^2\right)\right):\right)\exp\left(-:\tau H_{orbt}:\right)\cdot$$

$$\cdot\exp\left(-:\frac{\tau}{2}\left(\vec{W}^2\cdot\vec{s}+O\left(|\vec{s}|^2\right)\right):\right)\ =^k$$

$$=^k \ \exp\left(-:\tau H_{orbt}:\right)\exp\left(-:\tau\left(\vec{W}^3\cdot\vec{s}+O\left(|\vec{s}|^2\right)\right):\right) \ =^k$$

$$=^k \ \exp\left(-:\tau\left(\vec{W}^4\cdot\vec{s}+O\left(|\vec{s}|^2\right)\right):\right)\exp\left(-:\tau H_{orbt}:\right) \qquad (148)$$

Here τ is the size of the integration step and $=^k$ indicates that the differences between the right and left side have at least the order $O(|\ \tau\ |^{k+1})$.

Finally, let us select a decomposition formula from (148) and hence one of the vectors $\vec{W}^i(\vec{x}),\ i=1,2,3,4$. If we use the combination of some symplectic integration method of order k for orbital motion and the formulas (146), (147) for the mapping

$$\exp\left(:\lambda\left(\vec{W}^i(\vec{x},\tau)\cdot\vec{s}+O\left(|\vec{s}|^2\right)\right):\right)$$

($\lambda=\tau/2$ for $i=2$, or $\lambda=\tau$ for $i=1,3,4$) we will obtain a numerical method which has the order k, is symplectic for the orbital motion and automatically maintains the length of a spin vector equal to its initial value.

Example:

$$\vec{W}^1 \ = \ \vec{W} \ + \ \frac{\tau^2}{24}\cdot\left\{H_{orbt},\ \left\{H_{orbt},\ \vec{W}\right\}\right\} \ - \ \frac{\tau^2}{12}\cdot\left[\left\{H_{orbt},\ \vec{W}\right\}\times\vec{W}\right]$$

$$\vec{W}^2 \ = \ \vec{W} \ + \ \frac{\tau^2}{24}\cdot\left[\left\{H_{orbt},\ \vec{W}\right\}\times\vec{W}\right] \ - \ \frac{\tau^2}{12}\cdot\left\{H_{orbt},\ \left\{H_{orbt},\ \vec{W}\right\}\right\}$$

for $k=4$ and

$$\vec{W}^3 \ = \ \vec{W} \ + \ \frac{\tau}{2}\cdot\left\{H_{orbt},\ \vec{W}\right\}, \qquad \vec{W}^4 \ = \ \vec{W} \ - \ \frac{\tau}{2}\cdot\left\{H_{orbt},\ \vec{W}\right\}$$

for $k=2$.

Here the notation $\left\{H_{orbt},\ \vec{U}\right\}$ means the vector with components:

$$\left(\left\{H_{orbt},\ U_1\right\},\ \ \left\{H_{orbt},\ U_2\right\},\ \ \left\{H_{orbt},\ U_3\right\}\right)$$

9 Additional Transformations of the Spin-Orbit Hamiltonian

One of the most important problems in the design of accelerators is the problem of providing electric and magnetic fields which can hold a charged

particle beam in a sufficiently small neighbourhood of some geometrical line which we already introduced in subsection 4.3 under the name of closed design orbit[16]. One of the commonly used approaches to find a solution is to create electric and magnetic fields for which the given geometrical line will be the projection on R^3 of the six-dimensional trajectory of orbital motion in these fields, such that this trajectory has to be stable with respect to small perturbations of the initial conditions and of the values of electric and magnetic fields. We will now call this trajectory the **reference particle**, and all transformations of the spin-orbit Hamiltonian in this section will be more or less connected with this concept.

9.1 Closed Design Orbit and Reference Particle

Since in the final form of the equations we would like to keep the possibility to treat effects like misalignments of different electromagnetic elements of the accelerator, fluctuations in the values of electric and magnetic fields, influence of space charge, etc., we will assume that the reference particle is not a trajectory resulting from the Hamiltonian function (89), but is a solution of the system with Hamiltonian function

$$\tilde{H} \;=\; \tilde{H}_{orbt} + \vec{0}\cdot\vec{s} \;=\; -æx\tilde{\pi}_{\vec{B}} + æy\tilde{\pi}_{\vec{N}} - (1+hx+\alpha y)\tilde{\pi}_{\vec{T}} - \frac{e}{c}\tilde{A}_z \quad (149)$$

where

$$\tilde{\pi}_{\vec{N}} \;=\; P_x \;-\; \frac{e}{c}\,\tilde{A}_{\vec{N}}, \qquad \tilde{\pi}_{\vec{B}} \;=\; P_y \;-\; \frac{e}{c}\,\tilde{A}_{\vec{B}}$$

$$\tilde{\pi}_{\vec{T}} \;=\; \left(\frac{(E-e\tilde{\Phi})^2}{c^2} \;-\; m_0^2c^2 \;-\; \tilde{\pi}_{\vec{N}}^2 \;-\; \tilde{\pi}_{\vec{B}}^2\right)^{1/2}$$

and the components of the vector potential $\tilde{A}_{\vec{N}}$, $\tilde{A}_{\vec{B}}$, $\tilde{A}_z$, and the scalar potential $\tilde{\Phi}$, generally speaking, are not equal to the corresponding values

[16]In fact in this paper we never use the condition for the closed design orbit to be a **closed** curve, and so the equations derived can be used in applications to linear accelerators, cyclotrons and etc. Note also that in order to be more consistent in a coupled spin-orbit formalism the closed design orbit should be considered as a curve in the six-dimensional space of three orbital and three spin coordinates, but we do it in fact by assuming that for the spin part this curve satisfies the equation $\vec{s}=\vec{0}$ for all the times.

in the Hamiltonian function (89)[17].

Now we must discuss the conditions under which the functions

$$x_0(z), \quad P_x^0(z), \quad y_0(z), \quad P_y^0(z), \quad E_0(z), \quad t_0(z) \tag{150}$$

will be the solution of the equations of motion corresponding to the Hamiltonian (149), and the conditions under which the projection of this solution on R^3 will coincide with the closed design orbit.

Since x and y were introduced in subsection 4.3 as the transverse deviations from the closed design orbit it is necessary that

$$x_0(z) \equiv 0 \qquad \text{and} \qquad y_0(z) \equiv 0 \tag{151}$$

Substituting (150) and (151) in the equations

$$\begin{aligned} \frac{dx}{dz} &= \frac{\partial \tilde{H}_{orbt}}{\partial P_x} = \quad æy + \frac{(1+hx+\alpha y)}{\tilde{\pi}_{\vec{T}}}\,\tilde{\pi}_{\vec{N}} \\ \frac{dy}{dz} &= \frac{\partial \tilde{H}_{orbt}}{\partial P_y} = -æx + \frac{(1+hx+\alpha y)}{\tilde{\pi}_{\vec{T}}}\,\tilde{\pi}_{\vec{B}} \end{aligned}$$

we obtain that the momenta P_x^0 and P_y^0 have to be

$$P_x^0(z) = \frac{e}{c}\,\tilde{A}^0_{\vec{N}}(z) \qquad \text{and} \qquad P_y^0(z) = \frac{e}{c}\,\tilde{A}^0_{\vec{B}}(z) \tag{152}$$

where

$$\tilde{A}^0_{\vec{N}}(z) = \tilde{A}_{\vec{N}}(x_0(z),\, y_0(z),\, z,\, t_0(z))$$

$$\tilde{A}^0_{\vec{B}}(z) = \tilde{A}_{\vec{B}}(x_0(z),\, y_0(z),\, z,\, t_0(z))$$

Then putting (150)-(152) in the equations

$$\frac{dE}{dz} = \frac{\partial \tilde{H}_{orbt}}{\partial t}, \qquad \frac{dt}{dz} = -\frac{\partial \tilde{H}_{orbt}}{\partial E}$$

[17]Sometimes it is helpful to assume that even the charge and the rest mass of the reference particle are different from the corresponding values in the Hamiltonian function (89) too.

we obtain that $E_0(z)$ and $t_0(z)$ have to satisfy the equalities

$$\frac{dE_0}{dz} \;=\; -\frac{e}{c}\left(\frac{\partial \tilde{A}_z}{\partial t}\right)^0 \;+\; \frac{e}{\tilde{\pi}^0_{\vec{T}}}\cdot\frac{E_0 - e\tilde{\Phi}_0}{c^2}\cdot\left(\frac{\partial \tilde{\Phi}}{\partial t}\right)^0 \tag{153}$$

$$\frac{dt_0}{dz} \;=\; \frac{1}{\tilde{\pi}^0_{\vec{T}}}\,\frac{E_0 - e\tilde{\Phi}_0}{c^2} \tag{154}$$

where

$$\tilde{\Phi}_0(z) \;=\; \tilde{\Phi}(x_0(z),\, y_0(z),\, z,\, t_0(z)), \qquad \tilde{\pi}^0_{\vec{T}} \;=\; \sqrt{\frac{(E_0 - e\tilde{\Phi}_0)^2}{c^2} - m_0^2c^2}$$

$$\left(\frac{\partial \tilde{\Phi}}{\partial t}\right)^0 \;=\; \left.\frac{\partial \tilde{\Phi}}{\partial t}\right|_{\substack{x=x_0,y=y_0,\\ z=z,t=t_0}}, \qquad \left(\frac{\partial \tilde{A}_z}{\partial t}\right)^0 \;=\; \left.\frac{\partial \tilde{A}_z}{\partial t}\right|_{\substack{x=x_0,y=y_0,\\ z=z,t=t_0}}$$

The two remaining equations

$$\frac{dP_x}{dz} \;=\; -\frac{\partial \tilde{H}_{orbt}}{\partial x}, \qquad \frac{dP_y}{dz} \;=\; -\frac{\partial \tilde{H}_{orbt}}{\partial y}$$

give us two additional conditions which have to be satisfied too

$$\frac{e}{c}\,\frac{d\tilde{A}^0_{\vec{N}}}{dz} \;=\; h\tilde{\pi}^0_{\vec{T}} \;+\; \frac{e}{c}\left(\frac{\partial \tilde{A}_z}{\partial x}\right)^0 \;-\; \frac{e}{\tilde{\pi}^0_{\vec{T}}}\cdot\frac{E_0 - e\tilde{\Phi}_0}{c^2}\left(\frac{\partial \tilde{\Phi}}{\partial x}\right)^0 \tag{155}$$

$$\frac{e}{c}\frac{d\tilde{A}^0_{\vec{B}}}{dz} \;=\; \alpha\tilde{\pi}^0_{\vec{T}} \;+\; \frac{e}{c}\left(\frac{\partial \tilde{A}_z}{\partial y}\right)^0 \;-\; \frac{e}{\tilde{\pi}^0_{\vec{T}}}\frac{E_0 - e\tilde{\Phi}_0}{c^2}\left(\frac{\partial \tilde{\Phi}}{\partial y}\right)^0 \tag{156}$$

Here

$$\left(\frac{\partial \tilde{\Phi}}{\partial x}\right)^0 = \left.\frac{\partial \tilde{\Phi}}{\partial x}\right|_{\substack{x=x_0,y=y_0,\\ z=z,t=t_0}}, \qquad \left(\frac{\partial \tilde{\Phi}}{\partial y}\right)^0 = \left.\frac{\partial \tilde{\Phi}}{\partial y}\right|_{\substack{x=x_0,y=y_0,\\ z=z,t=t_0}}$$

$$\left(\frac{\partial \tilde{A}_z}{\partial x}\right)^0 = \left.\frac{\partial \tilde{A}_z}{\partial x}\right|_{\substack{x=x_0,y=y_0,\\ z=z,t=t_0}}, \qquad \left(\frac{\partial \tilde{A}_z}{\partial y}\right)^0 = \left.\frac{\partial \tilde{A}_z}{\partial y}\right|_{\substack{x=x_0,y=y_0,\\ z=z,t=t_0}}$$

So we find that the necessary and sufficient conditions for the functions (150) to be a reference particle with respect to a given closed design orbit and electromagnetic field are the conditions (151)-(156). These conditions (especially (155) and (156)) looks rather complicated, but in practice they can often be satisfied with the help of a very simple model of the closed design orbit (a sequence of line segments and arcs) and a corresponding piecewise constant model of the electromagnetic fields. We will not discuss in this paper the conditions of stability of this trajectory (and different definitions of stability) because this is one of the major problems of accelerator physics and it deserves special detailed consideration.

The reference particle is a trajectory, but for this trajectory just as for some real particle, we define quantities such as Lorentz factor γ_0, relative velocity β_0, and modulus of the velocity and kinetic momentum vectors, which we will denote by v_0 and π_0 respectively:

$$\gamma_0 \;=\; \frac{E_0 - e\tilde{\Phi}_0}{m_0 c^2}, \qquad \beta_0 \;=\; \sqrt{1 - \frac{1}{\gamma_0^2}}, \qquad v_0 \;=\; \beta_0 c$$

$$\pi_0 = m_0\gamma_0 v_0 = m_0\gamma_0\beta_0 c = m_0\gamma_0 c\sqrt{1-\frac{1}{\gamma_0^2}} = \sqrt{\frac{(E_0 - e\tilde{\Phi}_0)^2}{c^2} - m_0^2 c^2} = \tilde{\pi}^0_{\vec{T}}$$

In these new notations the equations (153) and (154) can be rewritten as

$$\frac{dE_0}{dz} \;=\; -\frac{e}{c}\left(\frac{\partial \tilde{A}_z}{\partial t}\right)^0 \;+\; \frac{e}{\beta_0 c}\left(\frac{\partial \tilde{\Phi}}{\partial t}\right)^0, \qquad \frac{dt_0}{dz} \;=\; \frac{1}{\beta_0 c} \;=\; \frac{1}{v_0} \qquad (157)$$

Note that sometimes it is more convenient to introduce the reference particle axiomatically just as an arbitrary set of six functions (150) satisfying only the conditions

$$x_0(z) \;\equiv\; 0, \qquad y_0(z) \;\equiv\; 0, \qquad \frac{dt_0}{dz} \;\neq\; 0$$

In this case we define the value of v_0 using the second equation in (157), and after this β_0, γ_0 and π_0 will be defined as follows

$$\beta_0 \;=\; \frac{v_0}{c}, \qquad \gamma_0 \;=\; \frac{1}{\sqrt{1-\beta_0^2}}, \qquad \pi_0 \;=\; m_0\,\gamma_0\,\beta_0\,c$$

In order, in the following, to have a uniform treatment for both ways of introducing the reference particle, for the axiomatic case we will define also $\tilde{A}^0_{\vec{N}}$, $\tilde{A}^0_{\vec{B}}$ and $\tilde{\Phi}_0$ as

$$\tilde{A}^0_{\vec{N}} \;=\; \frac{c}{e}P^0_x, \qquad \tilde{A}^0_{\vec{B}} \;=\; \frac{c}{e}P^0_y, \qquad \tilde{\Phi}_0 \;=\; \frac{E_0 - m_0\gamma_0 c^2}{e}$$

9.2 Deviations of Orbital Coordinates from the Solution for the Reference Particle

Since the reference particle is a reflection of our understanding (or of our desire) of how the accelerator should behave it is natural to introduce as new coordinates the deviations of orbit variables from the trajectory of the reference particle. Let us define new coordinates (x, y, σ) and new momenta (P_x, P_y, ε) as

$$\begin{aligned}
x_{new} &= x_{old}, & P_x^{new} &= P_x^{old} - P_x^0 \;=\; P_x^{old} - \tfrac{e}{c}\tilde{A}^0_{\vec{N}} \\
y_{new} &= y_{old}, & P_y^{new} &= P_y^{old} - P_y^0 \;=\; P_y^{old} - \tfrac{e}{c}\tilde{A}^0_{\vec{B}} \\
\sigma &= -(t - t_0)\,, & \varepsilon &= E \;-\; E_0
\end{aligned}$$

The orbit Hamiltonian in new variables has the form[18]

$$\hat{H}_{orbt} \;=\; -æ\, x\, \pi_{\vec{B}} \;+\; æ\, y\, \pi_{\vec{N}} \;-\; (1 + hx + \alpha y)\, \pi_{\vec{T}} \;-\; \frac{e}{c}A_z \;+$$

$$+\; \varepsilon\, \frac{dt_0}{dz} \;+\; \sigma\, \frac{dE_0}{dz} \;+\; \frac{e}{c}\left(x\, \frac{d\tilde{A}^0_{\vec{N}}}{dz} \;+\; y\, \frac{d\tilde{A}^0_{\vec{B}}}{dz} \right)$$

where now

$$\vec{\pi} \;=\; (\pi_{\vec{N}},\; \pi_{\vec{B}},\; \pi_{\vec{T}})$$

$$\pi_{\vec{N}} \;=\; P_x \;-\; \frac{e}{c}\,\Delta A_{\vec{N}}, \qquad \pi_{\vec{B}} \;=\; P_y \;-\; \frac{e}{c}\,\Delta A_{\vec{B}}$$

[18] Note, that before this transformation E was a generalized coordinate and t was a generalized momentum, but now we choose $\sigma = -(t - t_0)$ as the new generalized coordinate and $\varepsilon = E - E_0$ becomes a generalized momentum.

$$\pi_{\vec{T}} \;=\; \left(m_0^2c^2\left(\gamma^2-1\right) \;-\; \pi_{\vec{N}}^2 \;-\; \pi_{\vec{B}}^2\right)^{1/2}$$

$$\Delta A_{\vec{N}} \;=\; A_{\vec{N}} \;-\; \tilde{A}^0_{\vec{N}}, \qquad \Delta A_{\vec{B}} \;=\; A_{\vec{B}} \;-\; \tilde{A}^0_{\vec{B}}, \qquad \Delta\Phi \;=\; \Phi \;-\; \tilde{\Phi}_0$$

$$\gamma \;=\; \gamma_0 \;+\; \frac{\varepsilon - e\Delta\Phi}{m_0c^2}$$

and the spin part keeps the same form as in subsection 4.6 with the component of the vector $\vec{\pi}$ and γ defined above.

The variable σ describes the difference in arrival times at the position z between a given particle and the reference particle, and the quantity ε is their energy deviation. It seems to be more suitable to use another pair of canonical variables

$$\sigma_{new} \;=\; \beta_0\, c\, \sigma_{old} \qquad \text{and} \qquad \varepsilon_{new} \;=\; \frac{\varepsilon_{old}}{\beta_0 c}$$

where the new σ describes the longitudinal separation of the particle from the centre of the bunch. This canonical transformation does not concern any another variables and the new orbital part of the Hamiltonian becomes

$$\hat{H}_{orbt} \;=\; -æ\, x\, \pi_{\vec{B}} \;+\; æ\, y\, \pi_{\vec{N}} \;-\; (1+hx+\alpha y)\, \pi_{\vec{T}} \;-\; \frac{e}{c}A_z \;+$$

$$+\; \varepsilon \;+\; \frac{\sigma}{\beta_0 c}\,\frac{dE_0}{dz} \;+\; \frac{\sigma\varepsilon}{\pi_0\gamma_0^2}\frac{d\pi_0}{dz} \;+\; \frac{e}{c}\left(x\frac{d\tilde{A}^0_{\vec{N}}}{dz} \;+\; y\frac{d\tilde{A}^0_{\vec{B}}}{dz}\right)$$

whereas the spin part $\hat{H}_{spin}$ remains the same if we take into account the new expression for γ

$$\gamma \;=\; \gamma_0 \;+\; \frac{\beta_0}{m_0c}\left(\varepsilon - \frac{e}{\beta_0 c}\Delta\Phi\right)$$

9.3 Scaling of the Orbital Variables

The canonical scaling of the orbital variables which we use in this paper is

$$x_{new} \;=\; \sqrt{\pi_0}x_{old}, \qquad P_x^{new} \;=\; \frac{P_x^{old}}{\sqrt{\pi_0}}$$

$$y_{new} \;=\; \sqrt{\pi_0}\, y_{old}, \qquad P_y^{new} \;=\; \frac{P_y^{old}}{\sqrt{\pi_0}}$$

$$\sigma_{new} \;=\; \sqrt{\pi_0}\,\sigma_{old}, \qquad \varepsilon_{new} \;=\; \frac{\varepsilon_{old}}{\sqrt{\pi_0}}$$

This scaling is different from that usually used in accelerator physics (see Appendix E) and is applicable for both storage and acceleration regimes.

9.4 The General Form of the Spin-Orbit Hamiltonian in New Coordinates up to First Order with Respect to Spin Variables

Before writing out the final form of the spin-orbit Hamiltonian we would like to note that various authors use various coordinate systems for the treatment of fully coupled transverse and longitudinal motion (synchro-betatron motion). Our variables are closest in their physical meaning to the coordinates used in [30, 31].

In the variables introduced above the spin-orbit Hamiltonian takes the following final (at least for this paper) form

$$\hat{H} \;=\; \hat{H}_{orbt} \;+\; \hat{H}_{spin}$$

$$\hat{H}_{orbt} \;=\; -æx\left(\frac{\pi_{\vec{B}}}{\sqrt{\pi}_0}\right) \;+\; æy\left(\frac{\pi_{\vec{N}}}{\sqrt{\pi}_0}\right) \;-\; (\sqrt{\pi_0} + hx + \alpha y)\left(\frac{\pi_{\vec{T}}}{\sqrt{\pi_0}}\right) \;-\; \frac{e}{c}A_z \;+$$

$$+\; \sqrt{\pi_0}\,\varepsilon \;+\; \frac{\sigma}{\sqrt{\pi_0}\,\beta_0\, c}\,\frac{dE_0}{dz} \;+\; \frac{e}{\sqrt{\pi_0}\, c}\left(x\,\frac{d\tilde{A}^0_{\vec{N}}}{dz} \;+\; y\,\frac{d\tilde{A}^0_{\vec{B}}}{dz}\right) \;+$$

$$+\; \frac{1}{\pi_0}\left(\frac{\sigma\,\varepsilon}{\gamma_0^2} \;+\; \frac{x\,P_x \;+\; y\,P_y \;+\; \sigma\,\varepsilon}{2}\right)\frac{d\pi_0}{dz}$$

$$\hat{H}_{spin} \;=\; -\alpha\, s_x \;+\; h\, s_y \;-\; æ\, s_z \;+\; (\sqrt{\pi_0} \;+\; h\,x \;+\; \alpha\, y)\,\frac{m_0\,\gamma}{\sqrt{\pi_0}\,\pi_{\vec{T}}}\,\vec{W}\cdot\vec{s}$$

where

$$\vec{W} = -\frac{e}{m_0 \gamma c}\left((1+\gamma G)\,\vec{\mathcal{B}} - \frac{G\left(\vec{\pi}\cdot\vec{\mathcal{B}}\right)\vec{\pi}}{m_0^2 c^2 (1+\gamma)} - \frac{1}{m_0 c}\left(G + \frac{1}{1+\gamma}\right)\left[\vec{\pi}\times\vec{\mathcal{E}}\right]\right)$$

$$\vec{\pi} \;=\; (\pi_{\vec{N}},\; \pi_{\vec{B}},\; \pi_{\vec{T}})$$

$$\pi_{\vec{N}} \;=\; \sqrt{\pi_0}\left(P_x \;-\; \frac{e}{\sqrt{\pi_0}\,c}\,\Delta A_{\vec{N}}\right), \qquad \pi_{\vec{B}} \;=\; \sqrt{\pi_0}\left(P_y \;-\; \frac{e}{\sqrt{\pi_0}\,c}\,\Delta A_{\vec{B}}\right)$$

$$\pi_{\vec{T}} \;=\; \sqrt{m_0^2 c^2(\gamma^2-1) \;-\; \pi_{\vec{N}}^2 \;-\; \pi_{\vec{B}}^2}$$

$$\gamma \;=\; \gamma_0 \;+\; \frac{\sqrt{\pi_0}\,\beta_0}{m_0\,c}\left(\varepsilon \;-\; \frac{e}{\sqrt{\pi_0}\,\beta_0\,c}\,\Delta\Phi\right)$$

$$\gamma_0 \;=\; \frac{E_0 \;-\; e\,\tilde{\Phi}_0}{m_0\,c^2}, \qquad \pi_0 \;=\; m_0\,\gamma_0\,\beta_0\,c$$

Remembering that the electric and magnetic fields, and the vector and scalar potentials are supposed to be defined in the curvilinear coordinate system connected with the closed design orbit (see subsection 4.6 and appendix A for more details) and using the conversion formulae

$$\begin{aligned}
x' &= x/\sqrt{\pi_0}, & P'_x &= \sqrt{\pi_0}\,P_x \;+\; (e/c)\,\tilde{A}^0_{\vec{N}} \\
y' &= y/\sqrt{\pi_0}, & P'_y &= \sqrt{\pi_0}\,P_y \;+\; (e/c)\,\tilde{A}^0_{\vec{B}} \\
t' &= t_0 \;-\; \sigma\,/\left(\sqrt{\pi_0}\,\beta_0\,c\right), & E' &= E_0 \;+\; \left(\sqrt{\pi_0}\,\beta_0\,c\right)\varepsilon \\
z' &= z
\end{aligned}$$

from these variables, denoted here as z', x', P'_x, y', P'_y, E', t', to our final canonical coordinates z, x, P_x, y, P_y, σ, ε we obtain the rule for the substitution of the arguments of the mentioned above functions

$$F(t',\,x',\,y',\,z') \;\rightarrow\; F\left(t_0 - \frac{\sigma}{\sqrt{\pi_0}\beta_0 c},\;\; \frac{x}{\sqrt{\pi_0}},\;\; \frac{y}{\sqrt{\pi_0}},\;\; z\right)$$

In our final variables the relations between fields and potentials become:
The magnetic field:

$$\mathcal{B}_{\vec{N}} \;=\; \frac{\sqrt{\pi_0}}{\sqrt{\pi_0} \;+\; h\,x \;+\; \alpha\,y} \cdot \left(\sqrt{\pi_0}\,\frac{\partial A_z}{\partial y} \;-\; \frac{\partial A_{\vec{B}}}{\partial z} \;-\; \frac{æ\,y}{\sqrt{\pi_0}}\,\mathcal{B}_{\vec{T}} \right)$$

$$\mathcal{B}_{\vec{B}} \;=\; \frac{\sqrt{\pi_0}}{\sqrt{\pi_0} \;+\; h\,x \;+\; \alpha\,y} \cdot \left(\frac{\partial A_{\vec{N}}}{\partial z} \;-\; \sqrt{\pi_0}\,\frac{\partial A_z}{\partial x} \;+\; \frac{æ\,x}{\sqrt{\pi_0}}\,\mathcal{B}_{\vec{T}} \right)$$

$$\mathcal{B}_{\vec{T}} \;=\; \sqrt{\pi_0} \left(\frac{\partial A_{\vec{B}}}{\partial x} \;-\; \frac{\partial A_{\vec{N}}}{\partial y} \right)$$

where

$$A_z \;=\; \frac{1}{\sqrt{\pi_0}} \left((\sqrt{\pi_0} \;+\; h\,x \;+\; \alpha\,y)\; A_{\vec{T}} \;+\; æ\,(x\,A_{\vec{B}} \;-\; y\,A_{\vec{N}}) \right)$$

The electric field:

$$\mathcal{E}_{\vec{N}} \;=\; \sqrt{\pi_0} \left(\beta_0\,\frac{\partial A_{\vec{N}}}{\partial \sigma} \;-\; \frac{\partial \Phi}{\partial x} \right)$$

$$\mathcal{E}_{\vec{B}} \;=\; \sqrt{\pi_0} \left(\beta_0\,\frac{\partial A_{\vec{B}}}{\partial \sigma} \;-\; \frac{\partial \Phi}{\partial y} \right)$$

$$\mathcal{E}_{\vec{T}} \;=\; \sqrt{\pi_0} \left(\beta_0\,\frac{\partial A_{\vec{T}}}{\partial \sigma} \;-\; \frac{1}{\sqrt{\pi_0} \;+\; h\,x \;+\; \alpha\,y} \left(\frac{\partial \Phi}{\partial z} \;+\; æ \left(y\,\frac{\partial \Phi}{\partial x} \;-\; x\,\frac{\partial \Phi}{\partial y} \right) \right) \right)$$

To have the spin part of the Hamiltonian in more detailed form, let us introduce a vector $\vec{\Omega}$ by means of the equality $\hat{H}_{spin} = \vec{\Omega} \cdot \vec{s}$ and write out its components Ω_x, Ω_y and Ω_z:

$$\Omega_x \;=\; -\alpha \;+\; (\sqrt{\pi_0} \;+\; h\,x \;+\; \alpha\,y)\,\frac{e}{\sqrt{\pi_0}\,\pi_{\vec{T}}\,c} \cdot$$

$$\cdot \left[-(1 \;+\; \gamma\,G)\,\mathcal{B}_{\vec{N}} \;+\; \frac{G\,(\pi_{\vec{N}}\,\mathcal{B}_{\vec{N}} \;+\; \pi_{\vec{B}}\,\mathcal{B}_{\vec{B}} \;+\; \pi_{\vec{T}}\,\mathcal{B}_{\vec{T}})\,\pi_{\vec{N}}}{m_0^2\,c^2\,(1 \;+\; \gamma)} \;+ \right.$$

$$+ \frac{1}{m_0\, c} \left(G \, + \, \frac{1}{1 \, + \, \gamma} \right) (\pi_{\vec{B}}\, \mathcal{E}_{\vec{T}} \, - \, \pi_{\vec{T}}\, \mathcal{E}_{\vec{B}}) \Bigg]$$

$$\Omega_y \, = \, h \, + \, (\sqrt{\pi_0} \, + \, h\, x \, + \, \alpha\, y)\, \frac{e}{\sqrt{\pi_0}\, \pi_{\vec{T}}\, c} \cdot$$

$$\cdot \left[-(1 \, + \, \gamma\, G)\, \mathcal{B}_{\vec{B}} \, + \, \frac{G\, (\pi_{\vec{N}}\, \mathcal{B}_{\vec{N}} \, + \, \pi_{\vec{B}}\, \mathcal{B}_{\vec{B}} \, + \, \pi_{\vec{T}}\, \mathcal{B}_{\vec{T}})\, \pi_{\vec{B}}}{m_0^2 c^2 (1 + \gamma)} \, + \right.$$

$$\left. + \frac{1}{m_0\, c} \left(G \, + \, \frac{1}{1 \, + \, \gamma} \right) (\pi_{\vec{T}}\, \mathcal{E}_{\vec{N}} \, - \, \pi_{\vec{N}}\, \mathcal{E}_{\vec{T}}) \right]$$

$$\Omega_z \, = \, -æ \, + \, (\sqrt{\pi_0} \, + \, h\, x \, + \, \alpha\, y)\, \frac{e}{\sqrt{\pi_0}\, \pi_{\vec{T}}\, c} \cdot$$

$$\cdot \left[-(1 \, + \, \gamma\, G)\, \mathcal{B}_{\vec{T}} \, + \, \frac{G\, (\pi_{\vec{N}}\, \mathcal{B}_{\vec{N}} \, + \, \pi_{\vec{B}}\, \mathcal{B}_{\vec{B}} \, + \, \pi_{\vec{T}}\, \mathcal{B}_{\vec{T}})\, \pi_{\vec{T}}}{m_0^2\, c^2\, (1 \, + \, \gamma)} \, + \right.$$

$$\left. + \frac{1}{m_0\, c} \left(G \, + \, \frac{1}{1 \, + \, \gamma} \right) (\pi_{\vec{N}}\, \mathcal{E}_{\vec{B}} \, - \, \pi_{\vec{B}}\, \mathcal{E}_{\vec{N}}) \right]$$

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A How to Transform the Equations of the Electromagnetic Field to Curvilinear Coordinates Associated with the Closed Design Orbit

The purpose of this appendix is to point out the form which differential operators entering in the Maxwell system of equations for the electromagnetic field will have in the curvilinear coordinates considered (they could be equations for the vector and scalar potentials, or direct equations for the electric $\vec{\mathcal{E}}$ and magnetic $\vec{\mathcal{B}}$ fields)

In the coordinate system used every vector $\vec{A}$ can be uniquely represented in the form

$$\vec{A} \;=\; A_{\vec{N}} \cdot \vec{N} \;+\; A_{\vec{B}} \cdot \vec{B} \;+\; A_{\vec{T}} \cdot \vec{T}$$

(at least in a sufficiently small neighbourhood of closed design orbit).

As **physical Components** of $\vec{A}$ we understand the values

$$A_{\vec{N}} \;=\; A_{\vec{N}}(t,\, x,\, y,\, z), \quad A_{\vec{B}} \;=\; A_{\vec{B}}(t,\, x,\, y,\, z), \quad A_{\vec{T}} \;=\; A_{\vec{T}}(t,\, x,\, y,\, z)$$

Introduce the vectors

$$\vec{T}_z \;=\; (1 + hx + \alpha y)\, \vec{T} \;+\; æ \left(x\vec{B} \;-\; y\vec{N} \right)$$

$$\vec{N}_z \;=\; (1 + hx + \alpha y)\, \vec{N} \;+\; æ y \vec{T}$$

$$\vec{B}_z \;=\; (1 + hx + \alpha y)\, \vec{B} \;-\; æ x \vec{T}$$

and define A_z , A_x , and A_y as projections of the vector $\vec{A}$ on the vectors $\vec{T}_z$, $\vec{N}_z$ and $\vec{B}_z$ respectively

$$A_z \;=\; \vec{A} \cdot \vec{T}_z \,, \qquad A_x \;=\; \vec{A} \cdot \vec{N}_z \,, \qquad A_y \;=\; \vec{A} \cdot \vec{B}_z$$

Using the quantities introduced above we get the following formulae:

The Gradient of a Scalar Function:

$$\text{grad}\,\phi \;=\; \nabla\phi \;=\; \frac{1}{1+hx+\alpha y}\left[\frac{\partial\phi}{\partial z}\cdot\vec{T} \;+\; \frac{\partial\phi}{\partial x}\cdot\vec{N}_z \;+\; \frac{\partial\phi}{\partial y}\cdot\vec{B}_z\right]$$

The Laplacian of a Scalar Function:

$$\Delta\phi = \nabla^2\phi = \frac{1}{1+hx+\alpha y}\left[\frac{\partial}{\partial z}\left(\nabla\phi\cdot\vec{T}\right) + \frac{\partial}{\partial x}\left(\nabla\phi\cdot\vec{N}_z\right) + \frac{\partial}{\partial y}\left(\nabla\phi\cdot\vec{B}_z\right)\right]$$

The Divergence of a Vector Field:

$$\text{div}\vec{A} \;=\; \nabla\cdot\vec{A} \;=\; \frac{1}{1+hx+\alpha y}\left[\frac{\partial A_{\vec{T}}}{\partial z} \;+\; \frac{\partial A_x}{\partial x} \;+\; \frac{\partial A_y}{\partial y}\right]$$

The Curl of a Vector Field:

$$\text{curl}\vec{A} \;=\; \nabla\times\vec{A} \;=$$

$$=\; \frac{1}{1+hx+\alpha y}\left[\left(\frac{\partial A_{\vec{B}}}{\partial x} \;-\; \frac{\partial A_{\vec{N}}}{\partial y}\right)\vec{T}_z \;+\; \left(\frac{\partial A_z}{\partial y} \;-\; \frac{\partial A_{\vec{B}}}{\partial z}\right)\vec{N} +\right.$$

$$\left. +\; \left(\frac{\partial A_{\vec{N}}}{\partial z} \;-\; \frac{\partial A_z}{\partial x}\right)\vec{B}\right]$$

The Laplacian of a Vector Field:

The Laplacian of a vector field, if it is needed, can be expressed using the above formulae using the equality

$$\Delta\vec{A} \;=\; \nabla(\nabla\cdot\vec{A}) \;-\; \nabla\times(\nabla\times\vec{A})$$

B Simple Canonical Coordinates for the Periodic Solution

Let the differential equations of spin-orbit motion

$$\frac{d\vec{z}}{d\tau} \;=\; \{\vec{z},\, H(\tau,\, \vec{z})\}$$

T-periodic in τ and with possible nonlinear dependence of the Hamiltonian function $H(\tau,\, \vec{z}\,)$ on the variables $\vec{s}$ have the T-periodic solution $\vec{z}_*(\tau)$

$$\vec{x}(\tau) \;=\; \vec{x}_*(\tau), \qquad \vec{s}(\tau) \;=\; \vec{s}_*(\tau) \tag{158}$$

satisfying the condition

$$|\vec{s}_*(0)| \;\neq\; 0 \tag{159}$$

Since the parallel displacement $\vec{z}_{new} \;=\; \vec{z}_{old} \;-\; \vec{z}_*(\tau)$ which converts the solution (158) into the origin $\vec{0}$ is not a canonical transformation with respect to the coupled spin-orbit Poisson bracket if (159) holds, in this Appendix we introduce other simple canonical coordinates for the periodic solution (158).

Introduce the real skewsymmetric matrix

$$C(\tau,\, \vec{z}\,) \;=\; \begin{pmatrix} 0 & -\frac{\partial H}{\partial s_3} & \frac{\partial H}{\partial s_2} \\ \frac{\partial H}{\partial s_3} & 0 & -\frac{\partial H}{\partial s_1} \\ -\frac{\partial H}{\partial s_2} & \frac{\partial H}{\partial s_1} & 0 \end{pmatrix}$$

and write $\hat{C}(\tau) \;=\; C(\tau,\, \vec{z}_*(\tau))\,.$

It is easy to check that the unit vector

$$\vec{n}(\tau) \;=\; \frac{1}{|\vec{s}_*(0)|}\, \vec{s}_*(\tau)$$

periodically dependent on τ satisfies the equation

$$\frac{d\vec{n}}{d\tau} \;=\; \hat{C}(\tau)\, \vec{n} \tag{160}$$

Let us assume that we have found two unit vectors $\vec{m}(\tau)$ and $\vec{l}(\tau)$ T-periodic in τ which supplement the vector $\vec{n}(\tau)$ to form an orthogonal coordinate system satisfying the condition

$$\Big[\,\vec{m}(\tau)\times\vec{l}(\tau)\,\Big]\cdot\vec{n}(\tau)\;\equiv\;1 \tag{161}$$

Introduce new coordinates $\vec{y}$, $\vec{u}$ with the help of the canonical transformation

$$\vec{x}\;=\;\vec{x}_*(\tau)\;+\;\vec{y}\,,\qquad\qquad \vec{s}\;=\;A(\tau)\,\vec{u}$$

$$A(\tau)\;=\;\Big(\vec{m}(\tau),\,\vec{l}(\tau),\,\vec{n}(\tau)\Big)\;\in\;\mathrm{SO}(3)$$

In the variables $\vec{y},\;\vec{u}$ the Hamiltonian function takes the form

$$H_{new}=H_{old}\,(\tau,\;\vec{x}_*(\tau)+\vec{y},\;A(\tau)\vec{u})+\vec{y}\cdot J\frac{d\vec{x}_*}{d\tau}-\frac{1}{2}\vec{u}\cdot\mathrm{curl}_{\vec{u}}\left(A^\top\frac{dA}{d\tau}\right)$$

and the periodic solution $\vec{z}_*(\tau)$ is now expressed as follows

$$\vec{y}_*\;=\;\vec{0},\qquad\qquad \vec{u}_*\;=\;(0,\;0,\;|\vec{s}_*(0)|)$$

Now we wish to discuss the problem of existence and possible freedom of choice of the vectors $\vec{m}(\tau)$ and $\vec{l}(\tau)$ introduced above.

Taking the derivative with respect to τ in the identities

$$\vec{m}(\tau)\cdot\vec{n}(\tau)\;=\;0,\qquad\qquad \vec{m}(\tau)\cdot\vec{m}(\tau)\;=\;1$$

we have

$$\left(\frac{d\vec{m}}{d\tau}\;-\;\hat{C}(\tau)\vec{m}\right)\cdot\vec{n}\;=\;0\qquad\text{and}\qquad\frac{d\vec{m}}{d\tau}\cdot\vec{m}\;=\;0 \tag{162}$$

Subtracting the identity $\hat{C}\vec{m}\cdot\vec{m}\;=\;0$ from the second of the equalities (162) we find that the vector

$$\frac{d\vec{m}}{d\tau}\;-\;\hat{C}(\tau)\,\vec{m}$$

is orthogonal to the vectors $\vec{n}$ and $\vec{m}$ for all values of τ, and hence can be represented in the form

$$\frac{d\vec{m}}{d\tau}\;-\;\hat{C}(\tau)\,\vec{m}\;=\;\psi_1(\tau)\,\vec{l}$$

Similarly we have

$$\frac{d\vec{l}}{d\tau} \, - \, \hat{C}(\tau)\, \vec{l} \, = \, \psi_2(\tau)\, \vec{m}$$

From the condition

$$\frac{d}{d\tau}\left(\vec{m} \cdot \vec{l} \right) \, = \, 0$$

it follows that $\psi_2(\tau) \, = \, -\psi_1(\tau) \, \stackrel{\text{def}}{=} \, \psi(\tau)$, i.e. the vectors $\vec{m}, \, \vec{l}$ satisfy the system of differential equations

$$\frac{d\vec{m}}{d\tau} \, = \, \hat{C}(\tau)\vec{m} - \psi(\tau)\vec{l}, \qquad \frac{d\vec{l}}{d\tau} \, = \, \hat{C}(\tau)\vec{l} + \psi(\tau)\vec{m} \tag{163}$$

The solution of (163), written in complex notation, is

$$\vec{m}(\tau) + i\vec{l}(\tau) \, = \, \exp\left(i \int\limits_0^\tau \psi(\eta) d\eta \right) D(\tau)(\vec{m}(0) + i\vec{l}(0)) \tag{164}$$

where $D(\tau)$ is the fundamental matrix solution of (160) so that

$$\frac{dD}{d\tau} \, = \, \hat{C}(\tau)\, D, \qquad D(0) \, = \, I$$

Remembering that the vectors $\vec{m}$ and $\vec{l}$ must be T-periodic in τ we have from (164)

$$D(T)(\vec{m}(0) + i\vec{l}(0)) \, = \, \exp\left(-i \int\limits_0^T \psi(\eta) d\eta \right) (\vec{m}(0) + i\vec{l}(0)) \tag{165}$$

so that

$$\exp\left(-i \int\limits_0^T \psi(\eta) d\eta \right)$$

is an eigenvalue and $\vec{m}(0) + i\vec{l}(0)$ is the corresponding eigenvector of the matrix $D(T)$. Multiplying (165) onto the vector $\vec{m}(0) + i\vec{l}(0)$ we get

$$\exp\left(-i \int\limits_0^T \psi(\eta) d\eta \right) \, = \, \frac{1}{2} D(T)(\vec{m}(0) + i\vec{l}(0)) \cdot (\vec{m}(0) + i\vec{l}(0)) \tag{166}$$

This arguments can be reversed to show that the differentiable unit vectors $\vec{m}(\tau)$ and $\vec{l}(\tau)$ T-periodic in τ will supplement the vector $\vec{n}(\tau)$ to form an orthogonal basis satisfying the condition (161) if and only if they do so for $\tau = 0$ and then satisfy the differential equations (163) for some continuous function $\psi(\tau)$ for which the equality (166) holds.

Applying the transformation $\vec{s} = A(\tau)\,\vec{u}$ described above to the Hamiltonian

$$H \;=\; \vec{w}(\tau)\cdot\vec{s}, \qquad \vec{w}(\tau+2\pi) \;\equiv\; \vec{w}(\tau)$$

discussed in subsubsection 5.2.4 we find that in the variables $\vec{u}$ this Hamiltonian becomes

$$H \;=\; \psi(\tau)\cdot u_3$$

and is hence a normal form for all possible functions $\psi(\tau)$ satisfying (166), which are independent of τ.

C Sketch of the Proof of the Factorization Theorem

The purpose of this appendix is to prove the factorization theorem A.

The matrix of the quasi-linearization A_s is uniquely defined by the formula (111). Introduce maps

$$\mathcal{M}_2 \;=:A_s:, \quad \mathcal{M}_k \;=\; \mathcal{M}_{k-1}\exp(:F_k:), \quad k \;=\; 3,\ldots,m$$

Using induction in k, we will show that the functions $F_k \in \mathcal{H}_s(k)$ can be chosen in such a way that the conditions

$$\begin{cases} \mathcal{M}_k^{-1}\,\vec{X}(\vec{x},\,\vec{s}) \;=_{k+1}\; \vec{x} \;+\; \vec{X}_k(\vec{x},\,\vec{s}) \\ \\ \mathcal{M}_k^{-1}\,\vec{S}(\vec{x},\,\vec{s}) \;=_{k+2}\; \vec{s} \;+\; \vec{S}_{k+1}(\vec{x},\,\vec{s}) \end{cases} \tag{167}$$

are fulfilled where $\vec{X}_k \in \mathcal{H}_s(k)$ and $\vec{S}_{k+1} \in \mathcal{H}_s(k+1)$.

It is easy to see that the equality (167) for $k = m$ gives the proof of the theorem.

Obviously, (167) is correct at $k = 2$. Applying the operator

$$\exp(-:F_l(\vec{z}):)$$

to both parts of (167) for $k = l - 1$ we obtain that (167) will be correct for $k = l$, if the function F_l satisfies the equations

$$\{F_l,\, \vec{x}\} \;=\; \vec{X}_{l-1}, \qquad \{F_l,\, \vec{s}\} \;=\; \vec{S}_l \tag{168}$$

The structural matrix of the spin-orbit Poisson bracket has the form

$$\hat{J}(\vec{z}) \;=\; \mathrm{diag}\,(J,\, J_s(\vec{s}))$$

where the $2n \times 2n$ matrix J is the symplectic unit and

$$J_s(\vec{s}) \;=\; \begin{pmatrix} 0 & s_3 & -s_2 \\ -s_3 & 0 & s_1 \\ s_2 & -s_1 & 0 \end{pmatrix}$$

With the help of the matrix $\hat{J}$ the equations (168) can be written in the form of the system

$$J \cdot \mathrm{grad}_{\vec{x}}\, F_l \;=\; -\vec{X}_{l-1} \tag{169}$$

$$J_s(\vec{s}) \cdot \mathrm{grad}_{\vec{s}}\, F_l \;=\; -\vec{S}_l \tag{170}$$

Represent the vector function $\vec{X}_{l-1}$ in the form of a sum

$$\vec{X}_{l-1} \;=\; \sum_{k=0}^{\left[\frac{l-1}{2}\right]} \vec{X}^H_{l-2k-1}$$

where $\vec{X}^H_k(\vec{x},\, \vec{s})$ are homogeneous polynomials of degree k in the variables $\vec{x}$ and the symbol $[m]$ denotes the biggest integer which is smaller or equal to m.

Lemma 1: *Any solution $F_l(\vec{z}) \in \mathcal{H}_s(l)$ of the system (169)-(170) is given by the formula*

$$F_l \;=\; \vec{x} \cdot \sum_{k=0}^{\left[\frac{l-1}{2}\right]} \frac{1}{l-2k}\, J\, \vec{X}^H_{l-2k-1} \;-\; V(\vec{s})$$

where $V(\vec{s}) \in \mathcal{H}_s(l)$ satisfies the equation

$$J_s(\vec{s}) \cdot \mathrm{grad}_{\vec{s}}\, V \;=\; \vec{S}_l\left(\vec{0},\, \vec{s}\right) \tag{171}$$

The proof of this and following Lemmas is left as an exercise for the interested reader.

Starting from this point, we can forget about the existence of the orbital variables $\vec{x}$, because the proof of Theorem 1 has been reduced to finding the solution of equation (171), depending only on variables $\vec{s}$.

Let $\mathbf{H}_s(k)$ be the class of homogeneous polynomials of degree k in the variables $\vec{s}$. Our current task is to find the solution $V(\vec{s}) \in \mathbf{H}_s(k)$ of the equation

$$J_s(\vec{s}) \cdot \operatorname{grad} V \;=\; \vec{f}(\vec{s}) \tag{172}$$

where the vector function $\vec{f} \in \mathbf{H}_s(k),\;\; k \geq 1$ satisfies the condition

$$\operatorname{grad}\left(\vec{s} \cdot \vec{f}\right) \;=\; \operatorname{div}\left(\vec{f}\right) \cdot \vec{s} \tag{173}$$

The condition (173) follows from requiring that the map (110) is canonical.

Lemma 2: *If the solution $V(\vec{s}) \in \mathbf{H}_s(k)$ of the system (172) exists, then it satisfies the equation*

$$k(k+1)\, V \;-\; |\vec{s}|^2\, \Delta V \;=\; R(\vec{s}) \tag{174}$$

where

$$R(\vec{s}) \;=\; \vec{s} \cdot \mathit{curl}\, \vec{f}, \qquad \Delta \;=\; \frac{\partial^2}{\partial s_1^2} \;+\; \frac{\partial^2}{\partial s_2^2} \;+\; \frac{\partial^2}{\partial s_3^2}$$

Lemma 3: *If the solution of the equation (174) exists, then for odd k it is unique and for even k the difference between any two solutions is given by the formula*

$$V_1(\vec{s}) \;-\; V_2(\vec{s}) \;=\; c \cdot |\vec{s}|^k$$

where c is an arbitrary constant.

Lemma 4: *The equation (174) has a solution if and only if*

$$\Delta^{\left[\frac{k+1}{2}\right]} R(\vec{s}) \;=\; 0 \tag{175}$$

If (175) holds then the function

$$V \;=\; \sum_{j=0}^{\left[\frac{k-1}{2}\right]} a_j\, |\vec{s}|^{2j}\, \Delta^j\, R(\vec{s}) \tag{176}$$

where

$$a_j \;=\; \prod_{i=0}^{j} \frac{1}{(k-2i+1)(k-2i)}$$

satisfies the equation (174).

Lemma 5:

$$\Delta^j \left(\vec{s}\cdot \mathit{curl}\, \vec{f}\right) \;=\; \vec{s}\cdot \mathit{curl}\left(\Delta^j \vec{f}\right)$$

and therefore,

$$\Delta^{\left[\frac{k+1}{2}\right]} \left(\vec{s}\cdot \mathit{curl}\, \vec{f}\right) \;=\; 0$$

So in Lemmas 2 - 5 we have established that if the solution of the system (172) exists, then it is unique (up to an additive Casimir function of the spin-orbit Poisson bracket (18) $c\cdot|\vec{s}|^k$ at even k) and is given by the formula (176). Substituting (176) in (172), we obtain:

Lemma 6: *The necessary and sufficient conditions for the solvability of the equation (172) are that*

$$\vec{s}\cdot \vec{f} \;=\; 0 \qquad \textit{and} \qquad \mathit{div}\, \vec{f} \;=\; 0 \tag{177}$$

The complete solution of the problem is

Lemma 7: *If $\vec{f} \in \mathbf{H}_s(k)$, then conditions (177) and (173) are equivalent.*

Remark: From (167) it follows that the actual precision of the representation of spin variables (functions $\vec{S}(\vec{x},\, \vec{s}\,)$) in theorems 1 and 2 is higher by one order (in the sense of classes $\mathcal{O}_s(m)$) than the precision of the representation of orbital variables.

D How to Integrate Linear Equations of Spin Motion in Quadratures if We Know their Partial Solution

For general linear homogeneous systems of ordinary differential equations of order n knowledge of one nontrivial solution allows us to lower the order

of the system by one unit. So, in order to be able to find the general solution in quadratures we need to know $n-1$ linearly independent solutions. In this appendix we will show that due to a special symmetry, third order linear equations of spin motion can be integrated completely if we know only one partial solution.

Let the vector

$$\vec{n}(\tau) \;=\; (n_1(\tau),\, n_2(\tau),\, n_3(\tau)), \qquad |\vec{n}(\tau)| \;=\; 1$$

satisfy the system

$$\frac{d\vec{s}}{d\tau} \;=\; \vec{W}(\tau) \times \vec{s} \;=\; C\left(\vec{W}(\tau)\right)\vec{s} \tag{178}$$

Without loss of generality we will consider the τ-interval on which one of components of the vector $\vec{n}(\tau)$ is not equal to zero. Let it be $n_2(\tau)$. If on the considered interval $n_1(\tau) \neq 0$ (or $n_3(\tau) \neq 0$) we can begin by making a coordinate transformation $\vec{s}_{new} \;=\; B\vec{s}_{old}$ where the matrix

$$B \;=\; \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \left(\text{or } B \;=\; \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}\right)$$

Introduce the vectors

$$\vec{m}(\tau) \;=\; \frac{1}{\sqrt{1-n_3^2(\tau)}}\,(n_2(\tau),\, -n_1(\tau),\, 0)$$

$$\vec{l}(\tau) = \vec{n}(\tau) \times \vec{m}(\tau) = \frac{1}{\sqrt{1-n_3^2(\tau)}}(n_1(\tau)n_3(\tau),\, n_2(\tau)n_3(\tau),\, n_3^2(\tau)-1)$$

$$|\vec{m}(\tau)| \;=\; |\vec{l}(\tau)| \;=\; 1$$

and the new variables

$$\vec{u} \;=\; \begin{pmatrix} \vec{m}(\tau) \\ \vec{l}(\tau) \\ \vec{n}(\tau) \end{pmatrix} \cdot \vec{s} \;\stackrel{\text{def}}{=}\; A(\tau) \cdot \vec{s}$$

In the new variables the system (178) becomes

$$\frac{d\vec{u}}{d\tau} \;=\; \left(\frac{dA}{d\tau}\cdot A^{\top} \;+\; A\cdot C\left(\vec{W}\right)\cdot A^{\top}\right)\cdot\vec{u} \;\stackrel{\text{def}}{=}\; C\left(\vec{b}(\tau)\right)\cdot\vec{u} \qquad (179)$$

where the components of the vector $\vec{b}(\tau)$ are

$$b_1 \;=\; \frac{d\vec{n}}{d\tau}\cdot\vec{l} \;+\; \vec{n}\cdot C\left(\vec{W}\right)\vec{l}, \qquad b_2 \;=\; -\frac{d\vec{n}}{d\tau}\cdot\vec{m} \;-\; \vec{n}\cdot C\left(\vec{W}\right)\vec{m}$$

$$b_3 \;=\; \frac{d\vec{l}}{d\tau}\cdot\vec{m} \;+\; \vec{l}\cdot C\left(\vec{W}\right)\vec{m}$$

Since the vector $\vec{n}(\tau)$ satisfies the system (178), then $b_1(\tau)\equiv 0$ and $b_2(\tau)\equiv 0$. After some simple manipulations we get for $\omega = b_3$

$$\omega(\tau) \;=\; \frac{1}{1-n_3^2(\tau)}\left(W_1(\tau)\,n_1(\tau) \;+\; W_2(\tau)\,n_2(\tau)\right) \qquad (180)$$

and hence the solution of the system (179) is

$$\vec{u}(\tau) = \begin{pmatrix} \cos\Psi(\tau,\tau_0) & -\sin\Psi(\tau,\tau_0) & 0 \\ \sin\Psi(\tau,\tau_0) & \cos\Psi(\tau,\tau_0) & 0 \\ 0 & 0 & 1 \end{pmatrix}\cdot\vec{u}(\tau_0) \stackrel{\text{def}}{=} K(\tau,\tau_0)\cdot\vec{u}(\tau_0)$$

where

$$\Psi(\tau,\tau_0) \;=\; \int\limits_{\tau_0}^{\tau}\omega(\varepsilon)\,d\varepsilon$$

So the solution of the initial system (178) can be expressed now as follows

$$\vec{s}(\tau) \;=\; A^{\top}(\tau)\,K(\tau,\tau_0)\,A(\tau_0)\,\vec{s}(\tau_0)$$

where the matrix $A(\tau)$ has the form

$$A(\tau) \;=\; \begin{pmatrix} \frac{n_2(\tau)}{\sqrt{1-n_3^2(\tau)}} & -\frac{n_1(\tau)}{\sqrt{1-n_3^2(\tau)}} & 0 \\ \\ \frac{n_1(\tau)n_3(\tau)}{\sqrt{1-n_3^2(\tau)}} & \frac{n_2(\tau)n_3(\tau)}{\sqrt{1-n_3^2(\tau)}} & -\sqrt{1-n_3^2(\tau)} \\ \\ n_1(\tau) & n_2(\tau) & n_3(\tau) \end{pmatrix}$$

Remark: It is easy to see that in fact the procedure described above is applicable for condition weaker than $n_2(\tau) \neq 0$, $n_1^2(\tau) + n_2^2(\tau) \neq 0$ (which is equivalent to $|n_3(\tau)| < 1$).

Example: The equations of spin motion generated by the model Hamiltonian of the stationary single resonance problem

$$H(\tau, \vec{s}) \;=\; \varepsilon \cos(\Omega\tau + \phi) \cdot s_1 \;+\; \varepsilon \sin(\Omega\tau + \phi) \cdot s_2 \;+\; \lambda \cdot s_3 \qquad (181)$$

for $\varepsilon \neq 0$ have the nontrivial $(2\pi/\Omega)$-periodic solution

$$\vec{n}(\tau) \;=\; \frac{1}{\sqrt{\varepsilon^2 + (\lambda - \Omega)^2}} \left(\varepsilon \cos(\Omega\tau + \phi),\;\; \varepsilon \sin(\Omega\tau + \phi),\;\; \lambda - \Omega\right)$$

which satisfies the condition $|n_3(\tau)| < 1$ for all values of τ.

The $(2\pi/\Omega)$-periodic matrix $A(\tau)$ defined by this solution has the form

$$A(\tau) \;=\; \begin{pmatrix} \frac{\varepsilon}{|\varepsilon|} \sin(\Omega\tau + \phi) & -\frac{\varepsilon}{|\varepsilon|} \cos(\Omega\tau + \phi) & 0 \\ \frac{\varepsilon}{|\varepsilon|} \frac{\lambda - \Omega}{\omega} \cos(\Omega\tau + \phi) & \frac{\varepsilon}{|\varepsilon|} \frac{\lambda - \Omega}{\omega} \sin(\Omega\tau + \phi) & -\frac{|\varepsilon|}{\omega} \\ \frac{\varepsilon}{\omega} \cos(\Omega\tau + \phi) & \frac{\varepsilon}{\omega} \sin(\Omega\tau + \phi) & \frac{\lambda - \Omega}{\omega} \end{pmatrix}$$

where ω, calculated in accordance with (180), is

$$\omega \;=\; \sqrt{\varepsilon^2 + (\lambda - \Omega)^2} \;=\; const$$

So the fundamental matrix solution of the stationary single resonance problem

$$M(\tau, \tau_0) \;=\; A^{\top}(\tau)\, K(\tau, \tau_0)\, A(\tau_0)$$

expressed in terms of matrix elements m_{ij} takes the following form

$$\begin{cases} m_{11} = \alpha \sin(\Omega\tau_0 + \phi) + \left(\frac{\lambda-\Omega}{\omega}\beta + \left(\frac{\varepsilon}{\omega}\right)^2 \cos(\Omega\tau + \phi)\right) \cos(\Omega\tau_0 + \phi) \\ m_{12} = \left(\frac{\lambda-\Omega}{\omega}\beta + \left(\frac{\varepsilon}{\omega}\right)^2 \cos(\Omega\tau + \phi)\right) \sin(\Omega\tau_0 + \phi) - \alpha \cos(\Omega\tau_0 + \phi) \\ m_{13} = \frac{\varepsilon}{\omega}\left(\frac{\lambda-\Omega}{\omega}\cos(\Omega\tau + \phi) - \beta\right) \\ m_{21} = \zeta \sin(\Omega\tau_0 + \phi) + \left(\frac{\lambda-\Omega}{\omega}\gamma + \left(\frac{\varepsilon}{\omega}\right)^2 \sin(\Omega\tau + \phi)\right) \cos(\Omega\tau_0 + \phi) \\ m_{22} = \left(\frac{\lambda-\Omega}{\omega}\gamma + \left(\frac{\varepsilon}{\omega}\right)^2 \sin(\Omega\tau + \phi)\right) \sin(\Omega\tau_0 + \phi) - \zeta \cos(\Omega\tau_0 + \phi) \\ m_{23} = \frac{\varepsilon}{\omega}\left(\frac{\lambda-\Omega}{\omega}\sin(\Omega\tau + \phi) - \gamma\right) \\ m_{31} = \frac{\varepsilon}{\omega}\left(\vartheta \cos(\Omega\tau_0 + \phi) - \sin(\omega(\tau - \tau_0)) \sin(\Omega\tau_0 + \phi)\right) \\ m_{32} = \frac{\varepsilon}{\omega}\left(\sin(\omega(\tau - \tau_0)) \cos(\Omega\tau_0 + \phi) + \vartheta \sin(\Omega\tau_0 + \phi)\right) \\ m_{33} = \left(\frac{\varepsilon}{\omega}\right)^2 \cos(\omega(\tau - \tau_0)) + \left(\frac{\lambda-\Omega}{\omega}\right)^2 \end{cases}$$

where we have used the notations

$$\begin{cases} \alpha = \cos(\omega(\tau - \tau_0)) \sin(\Omega\tau + \phi) + \frac{\lambda-\Omega}{\omega} \sin(\omega(\tau - \tau_0)) \cos(\Omega\tau + \phi) \\ \beta = \frac{\lambda-\Omega}{\omega} \cos(\omega(\tau - \tau_0)) \cos(\Omega\tau + \phi) - \sin(\omega(\tau - \tau_0)) \sin(\Omega\tau + \phi) \\ \gamma = \sin(\omega(\tau - \tau_0)) \cos(\Omega\tau + \phi) + \frac{\lambda-\Omega}{\omega} \cos(\omega(\tau - \tau_0)) \sin(\Omega\tau + \phi) \\ \zeta = \frac{\lambda-\Omega}{\omega} \sin(\omega(\tau - \tau_0)) \sin(\Omega\tau + \phi) - \cos(\omega(\tau - \tau_0)) \cos(\Omega\tau + \phi) \\ \vartheta = \frac{\lambda-\Omega}{\omega}\left(1 - \cos(\omega(\tau - \tau_0))\right) \end{cases}$$

E Noncanonical Scaling of the Orbital Variables in the Case of the Simplest Storage Regime

In the case of the simplest storage regime, that is

$$\pi_0(z) \equiv const > 0,$$

it is typical in accelerators physics to keep the orbital position variables unchanged and to normalize the corresponding momenta to the value of the design kinetic momentum π_0. So, following tradition, we would like to have the possibility to describe the orbital motion using the new variables

$$\begin{cases} x_{new} = x_{old}, & y_{new} = y_{old}, & \sigma_{new} = \sigma_{old} \\ P_x^{new} = P_x^{old}/\pi_0, & P_y^{new} = P_y^{old}/\pi_0, & \varepsilon_{new} = \varepsilon_{old}/\pi_0 \end{cases} \tag{182}$$

where the superscript *'old'* indicates variables of subsection 9.2.

Unfortunately the coordinate transformation (182) is not symplectic and even if for the study of the orbital motion alone we can treat this transformation as canonical using a rescaled Hamiltonian (see remark 2 in subsection 4.1), we cannot proceed in the same manner for the case of fully coupled equations of spin-orbit motion. So we will restrict ourselves to presenting the Hamiltonian which will give us the correct form of the triangular system in new variables. This will not be applicable to the study of the effect of the spin on the orbit motion if we admit the complete equations of motion (21)-(23).

In the variables introduced above the spin-orbit Hamiltonian becomes

$$\breve{H} = \breve{H}_{orbt} + \breve{H}_{spin}$$

$$\breve{H}_{orbt} = \frac{\hat{H}_{orbt}}{\pi_0} = -æ\, x\, \breve{\pi}_{\vec{B}} + æ\, y\, \breve{\pi}_{\vec{N}} - (1 + hx + \alpha y)\, \breve{\pi}_{\vec{T}} -$$

$$- \frac{e}{\pi_0 c} A_z + \varepsilon + \frac{\sigma}{\pi_0 \beta_0 c} \frac{dE_0}{dz} + \frac{e}{\pi_0 c} \left(x \frac{d\tilde{A}^0_{\vec{N}}}{dz} + y \frac{d\tilde{A}^0_{\vec{B}}}{dz} \right)$$

$$\breve{H}_{spin} \;=\; -\alpha\, s_x \;+\; h\, s_y \;-\; æ\, s_z \;+\; \frac{(1+hx+\alpha y)}{\breve{\pi}_{\vec{T}}}\, \vec{\breve{W}} \cdot \vec{s}$$

where

$$\vec{\breve{W}} \;=\; -\frac{e}{\pi_0 c}\left((1+\gamma G)\,\vec{\mathcal{B}} \;-\; \frac{\pi_0^2 G\left(\vec{\breve{\pi}}\cdot\vec{\mathcal{B}}\right)\vec{\breve{\pi}}}{m_0^2 c^2 (1+\gamma)} \;-\; \frac{\pi_0}{m_0 c}\left(G+\frac{1}{1+\gamma}\right)\left[\vec{\breve{\pi}}\times\vec{\mathcal{E}}\right]\right)$$

$$\vec{\breve{\pi}} = (\breve{\pi}_{\vec{N}},\; \breve{\pi}_{\vec{B}},\; \breve{\pi}_{\vec{T}})$$

$$\breve{\pi}_{\vec{N}} \;=\; \frac{\pi_{\vec{N}}}{\pi_0} \;=\; P_x \;-\; \frac{e}{\pi_0 c}\,\Delta A_{\vec{N}}, \qquad \breve{\pi}_{\vec{B}} \;=\; \frac{\pi_{\vec{B}}}{\pi_0} \;=\; P_y \;-\; \frac{e}{\pi_0 c}\,\Delta A_{\vec{B}}$$

$$\breve{\pi}_{\vec{T}} \;=\; \frac{\pi_{\vec{T}}}{\pi_0} \;=\; \sqrt{\frac{\gamma^2-1}{\beta_0^2\gamma_0^2} \;-\; \breve{\pi}_{\vec{N}}^2 \;-\; \breve{\pi}_{\vec{B}}^2}$$

$$\gamma \;=\; \gamma_0 \;+\; \gamma_0\beta_0^2\left(\varepsilon \;-\; \frac{e}{\pi_0\beta_0 c}\,\Delta\Phi\right)$$

For such scaling the conversion formulae between variables in the curvilinear coordinate system connected with the closed orbit z', x', P'_x, y', P'_y, E', t' and our final coordinates z, x, P_x, y, P_y, σ, ε have the form

$$\begin{aligned}
x' &= x, & P'_x &= \pi_0 P_x \;+\; (e/c)\tilde{A}^0_{\vec{N}} \\
y' &= y, & P'_y &= \pi_0 P_y \;+\; (e/c)\tilde{A}^0_{\vec{B}} \\
t' &= t_0 \;-\; \sigma/(\beta_0 c), & E' &= E_0 \;+\; (\pi_0\beta_0 c)\,\varepsilon \\
z' &= z
\end{aligned}$$

The rule for the substitution of the arguments reads now as

$$F(t',\, x',\, y',\, z') \;\rightarrow\; F\left(t_0 - \frac{\sigma}{\beta_0 c},\;\; x,\;\; y,\;\; z\right)$$

The expressions for the projections of the magnetic field and for A_z keep the same form as in subsection 4.6, and the connection between the electric field and the potentials becomes

$$\mathcal{E}_{\vec{N}} \;=\; -\frac{\partial\Phi}{\partial x} \;+\; \beta_0\,\frac{\partial A_{\vec{N}}}{\partial\sigma}$$

$$\mathcal{E}_{\vec{B}} \;=\; -\frac{\partial\Phi}{\partial y} \;+\; \beta_0\,\frac{\partial A_{\vec{B}}}{\partial\sigma}$$

$$\mathcal{E}_{\vec{T}} \;=\; -\frac{1}{1+hx+\alpha y}\left(\frac{\partial\Phi}{\partial z} \;+\; æ\left(y\,\frac{\partial\Phi}{\partial x} \;-\; x\,\frac{\partial\Phi}{\partial y}\right)\right) \;+\; \beta_0\,\frac{\partial A_{\vec{T}}}{\partial\sigma}$$

F FORGET-ME-NOT, a Computer Code for the Study of Polarized Beam Dynamics

The computer code **FORGET-ME-NOT** has been written for the study of unpolarized and polarized beam dynamics and among other things includes the following important options:

1. Calculation of the strengths of the imperfection spin resonances and first order intrinsic spin resonances with betatron oscillations with the help of an averaging method.

2. Calculation of one-turn Taylor maps for orbit and spin motion up to arbitrarily high order with respect to the amplitudes of the betatron and synchrotron oscillations and determination of

2.1. Invariant functions of the orbit motion.

2.2. Equilibrium polarization direction.

2.3. Dependence of orbit and spin tunes on the invariants of orbit motion (spread of orbit and spin tunes).

3. Numerical tracking of particles with spin in accelerators and storage rings preserving:

3.1. Symplecticity with respect to the 6-D orbit motion.

3.2. Orthogonality with respect to the 3-D spin motion.

All options use the same physical model. The use of various approaches allows us to understand the computed results from various points of view. FORGET-ME-NOT, for example, has been applied to the investigation of

schemes for preserving the polarization in the TRIUMF KAON Booster [6, 7], to the investigation of spin motion at high energies in the HERA proton ring [11] and to the study of the possibility to accelerate the polarized proton beam in the Nuclotron ring in Dubna [32].

References

[1] Balandin, V.V., Golubeva, N.I., *Nonlinear Spin Dynamics*, Proc. XV Int. Conf. High Energy Accelerators, Hamburg, 1992, Int. J. Mod. Phys. A, **2B**, 998, (1992).

[2] Balandin, V.V., Golubeva, N.I., *Hamiltonian Methods for the Study of Polarized Proton Beam Dynamics in Accelerators and Storage Rings*, Computational Accelerator Physics Conference, Williamsburg, Virginia, USA, (1996).

[3] Thomas, L.H., *Philos. Mag.,* **3**:1, (1927).

[4] Bargmann, V., Michel, L., and Telegdi, V.L., *Precession of the polarization of particles moving in a homogeneous electromagnetic field*, Phys. Rev. Lett., 2(10):435-436, (1959).

[5] Landay, L.D., Lifshitz, E.M., *The Classical Theory of Fields*, Volume 2 of Course of Theoretical Physics, London-Paris, 1959.

[6] Balandin, V.V., Golubeva, N.I., *Investigation of Spin Motion in the Booster Lattice*, TRIUMF Report TRI-DN-93-K236, (1993).

[7] Balandin, V.V., Golubeva, N.I., *Fast Betatron Tune Jumps and Partial Siberian Snakes for Preserving the Polarization in the Booster Lattice*, TRIUMF Report TRI-DN-93-K240, (1993).

[8] Balandin, V.V., Golubeva, N.I., *Taylor Maps and Calculations of Equilibrium Polarization Direction for Proton Storage Rings*, Particle Accelerator Conference, Washington, USA, (1993).

[9] Balandin, V.V., Golubeva, N.I., *Investigation of Spin Resonance Crossing in Proton Circular Accelerators*, Particle Accelerator Conference, Washington, USA, (1993).

[10] Balandin, V.V., *One of Methods to Extract Truncated Taylor Map for Orbital and Spin Motion in Proton Storage Rings*, Particle Accelerator Conference, Washington, USA, (1993).

[11] Balandin, V.V., Golubeva, N.I., Barber, D.P., *Studies of the Behaviour of Proton Spin Motion in HERA-p at High Energies*, DESY M 96-04, (1996).

[12] Guillemin V., Sternberg S., *Symplectic Techniques in Physics*, Cambridge University Press, 1984.

[13] Nemytskij, V.V., Stepanov, V.V. *Qualitative theory of differential equations*, Moscow-Leningrad: OGIZ 1947. (English translations: Princeton University Press, 1960.)

[14] Dubrovin, B.A., Fomenko, A.T., Novikov, S.P., *Modern Geometry - Methods and Applications*, Nauka, Moscow 1979. (English translations in two parts: Part 1: *The geometry of surfaces, transformation groups and fields*, Graduate Texts in Mathematics 93, Springer-Verlag, 1984; Part 2: *The geometry and topology of manifolds*, Graduate Texts in Mathematics 104, Springer-Verlag, 1985.)

[15] Olver, P.J., *Applications of Lie Groups to Differential Equations*, Second Edition, Graduate Texts in Mathematics 107, Springer-Verlag, 1993.

[16] Montague, B.W., *Elementary Spinor Algebra for Polarized Beams in Storage Rings*, Particle Accelerators, Vol. 11, (1981).

[17] Cartan, E. *The Theory of Spinors*. HERMAMN, Paris, 1966.

[18] Dragt, A.J., Finn, J.M., *Lie Series and Invariant Functions for Analytic Symplectic maps*, J.Math. Phys, **17**, (1976).

[19] Dragt, A.J., *Lectures on nonlinear Orbit Dynamics*, in *Physics of High Energy Particle Accelerators*, AIP Conf. Proc. 87, (1982).

[20] Bruno, A.D., *The restricted* 3*-Body Problem*. Nauka, Moscow 1986. (English translations in: de Gruyter Expositions in Mathematics 17, Walter de Gruyter, 1994.)

[21] Yakubovich, V.A., Starzhinskii, *Linear differential equations with periodic coefficients and their applications*. Nauka, Moscow 1972. (English translations in two parts: Vol. 1,2, Israel Program for Scientific Translations, Jerusalem, and Wiley, New York, 1975.)

[22] Marsden, J.E., *Lectures on Mechanics*, Cambridge University Press, 1992.

[23] Quispel, G.R.W., Capel, H.W., *Solving ODEs numerically while preserving a first integral* , Phys. Lett. **A 218**, 223, (1996).

[24] Suzuki, M., *Fractal decomposition of exponential operators with applications to many-body theories and Monte Carlo simulations*, Phys. Lett. **A 146**, 319, (1990).

[25] Suzuki, M., *General theory of higher-order decomposition of exponential operators and symplectic integrators*, Phys. Lett. **A 165**, 387, (1992).

[26] Sanz-Serna, J.M., Calvo, M.P., *Numerical Hamiltonian Problem*, London; New-York: Chapman and Hall, 1994.

[27] Yoshida, H., *Construction of Higher Order Symplectic Integrators*, Phys. Lett. **A 150**, 262, (1990).

[28] Forest, E., Bengtsson, J., Reusch, M.F., *Application of the Yoshida-Ruth techniques to implicit integration and multi-map explicit integration*, Phys. Lett. **A 158**, 99, (1991).

[29] Levich, Benjamin G., *Theoretical Physics*, Volume 1: *Theory of the Electromagnetic Field. Theory of Relativity*, Amsterdam-London, 1970.

[30] Mais, H., Ripken, G., *Theory of Coupled Synchro-Betatron Oscillations (I)*, DESY M-82-05 and later papers.

[31] Barber, D.P., Mais, H., Ripken, G., Willike, F., *Nonlinear Theory of Coupled Synchro-Betatron Motion*, DESY 86-147 and later papers.

[32] Kondratenko, A.M., Golubeva, N.I., *Depolarization of Proton Beam in the Nuclotron Ring at JINR*, VII Workshop on High Energy Spin Physics, Dubna, July 1997.

[33] a) Derbenev, Ya.S., *How the Stern-Gerlach effect could work for polarization in storage rings?*, University of Michigan, Ann Arbor, Preprint UM HE 90-23, 1990.
b) Derbenev, Ya.S., *The Stern-Gerlach method in charged particle storage rings. - 1. The double-resonance Stern-Gerlach effect*, University of Michigan, Ann Arbor, Preprint UM HE 90-30, 1990.
c) Derbenev, Ya.S., *The Stern-Gerlach method in charged particle storage rings. - 2. The neo-classical Stern-Gerlach effect*, University of Michigan, Ann Arbor, Preprint UM HE 90-32, 1990.

[34] Niinikoski, T.O., Rossmanith, R., Nucl.Inst.Meth., A255, 469(1987).

[35] Barber, D.P., Heinemann, K., Ripken, G., *A canonical* 8*-dimensional formalism for classical spin-orbit motion in storage rings. I. A new pair of canonical variables*, Z. Phys. C65 117-142(1994).